Why study algebras over functors?

Algebras, Monads, and the proof of Beck's monadicity theorem

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In this essay we present a development of the theory of monadic functors, naturally motivated first by examples in elementary abstract algebra, and then for understanding algebras over monads for their own sake. At every stage, we present a development which persuades the reader that they could have obtained our results themselves, by combining some inspiration with a propensity for "following their nose". Indeed, we stumble into discovering the special coequalisers needed in the famously unintuitive hypotheses of Beck's monadicity theorem—our main result. A proof of the this theorem is the ultimate payoff for our investment, which is given in modern, "ethical[1]", terms.

During the moments where a greater leap of creativity is required, we have structured the presentation so that beautiful symmetries in the underlying structures are brought to the fore (a good example is the duality between the preservation and reflection of coequalisers, and whether the unit or counit of a particular adjunction is invertible). Perhaps then at least the reader may identify similar patterns more easily in other contexts.

Beck's theorem has wide-ranging implications, from classifying when algebras over endofunctors are the same as algebras over monads, through the apparent connection between algebraic structures to monads (related to Lawvere theory), to the theory of descent in modern algebraic geometry and sheaf theory. Indeed, the faithfully flat descent of Grothendieck's famous FGA (and other works) is a special case of Beck's theorem, and the more modern work of Deligne on Tannakian categories exploits Beck's theorem to place the theory on more elegant foundations.

1 Algebras over endofunctors

We begin with a humble task; we seek to encode the idea of the many possible algebraic structures, such as a monoids, or groups, or rings, in a category theoretic language. First, we have in mind some fixed category of discourse $\mathscr{C}$ (for concreteness one could take $\mathscr{C} = \mathsf{Sets}$, but there is no reason to prefer this category over $\mathsf{Monoids}$, or even $\mathsf{DiffMan}$!). We might claim that every algebraic structure on $\mathscr{C}$ is fundamentally the data of how to take some collection of symbols "contained" in an object $C \in \mathscr{C}$, and combine them. If $T(C)$ is an object of $\mathscr{C}$ which collects for us all possible ways of forming combinations of elements of C, then the operations on our algebraic structure could reasonably be the data of a (potentially quite complicated) morphism $f : T(C) \to C$. Of course, if our structure is to be at all general, a rule (morphism) for changing from one collection of symbols C to another collection C' should be compatible with taking all combinations $T(C)$ to those in $T(C')$; therefore, $T : \mathscr{C} \to \mathscr{C}$ should be a functor. We will call a pair (C, f) defining an instance of an algebraic structure of type or *signature* T, an algebra over T. These considerations motivate the following formal definitions.

Definition 1.1. Given an arbitrary endofunctor $T : \mathscr{C} \to \mathscr{C}$, we call a pair $(C \in \mathscr{C}, f : T(C) \to C)$ an *algebra over the endofunctor* T, or more concisely, a T*-algebra*. In this case, C is called the *carrier* of the algebra, and the $f : T(C) \to C$ is called the *structure map*.

A morphism of T-algebras $k : (C, f) \to (C', f')$ is a morphism in the original category $\mathscr{C}$ which is compatible with the algebra structure, in that the diagram

$$\begin{tikzcd} T(C) \arrow[r, "Tk"] \arrow[d, "f"'] & T(C') \arrow[d, "f'"] \\ C \arrow[r, "k"] & C' \end{tikzcd}$$

is commutative.

[1] As opposed to the *evil* version of Mac Lane's *Categories for the Working Mathematician* [11], for example.

Because we can stack the diagram defining algebra morphisms side-to-side, the composition of such morphisms is still an algebra morphism. Furthermore, we have identities, and composition is associative because composition in the original category is. Therefore, to each endofunctor $T : \mathscr{C} \to \mathscr{C}$ is associated a category $\mathbf{Alg}(T)$ of T-algebras.

In order to test the utility of our construction, we will work in $\mathscr{C} = \mathsf{Sets}$ and attempt to encode the idea of a group. Indeed, every group G can be thought of an underlying set of the same name, equipped with

- a multiplication operation $\cdot : G \times G \to G$,
- an inversion operation ${}^{-1} : G \to G$, and
- an inverse element (the morphism) $e : \bullet \to G$ (with $\bullet$ the 1-element set).

Together these three pieces of data define a morphism of sets $G \times G \sqcup G \sqcup \bullet \to G$ (with "$\sqcup$" the disjoint union of sets), and so we are lead to define a functor $T : \mathsf{Sets} \to \mathsf{Sets}$ on objects X by $T(X) = X \times X \sqcup X \sqcup \bullet \to X$.

Every T-algebra for T defined in this way is thus a candidate for a structure of a group. However, decoding arbitrary structure maps for these T-algebras need not lead to actual group structures—for any algebraic structure, there are some number of identities, or relations, which we must ensure are maintained. This cuts down $\mathbf{Alg}(T)$ to some subcategory, but this is annoying—our implementation of relations in our algebraic structures is entirely ad-hoc. There are some redeeming features however; for example, if (C, f) and (C', f') are actually honest group structures, then an algebra morphism between them *is* a homomorphism of groups! Indeed, the theory of algebras over endofunctors already possesses a rich theory.[2]

Of greater importance to us is the fact that all of the algebraic structures which we can quickly think of share a few common operations; for one thing, every element of the carrier $C \in \mathscr{C}$ can be included in a canonical way into the collection of combinations of all such elements $T(C)$ (this is just the "trivial combination", consisting of a single element). We would like a generic way into express this additional structure so that it is accessible to the mathematics.

2 Algebras over monads

The choice of extra structure with which to equip endofunctors $T : \mathscr{C} \to \mathscr{C}$ in order to address the considerations of the previous section are a matter of taste. One possibility is to ask first for a family of morphisms $\eta_C : C \to T(C)$ which send each element of an object $C \in \mathscr{C}$ to the "trivial combination" consisting of that element in $T(C)$, and do so in a way which is natural with respect to T. Without knowing the underlying implementation of the "operations" f of an instance (C, f) of an algebraic structure T, we will also require that for fixed T there is a canonical way of resolving "combinations of combinations of elements of C" into only "combinations of elements of C" (and again we would like this to be natural with respect to T). Thus, we also desire a natural transformation $\mu : T^2 \to T$ associated with T.

The structure we have just informally described is called a *monad*, and we take this opportunity to introduce the formal definitions.

Definition 2.1. A *monad* (T, η, μ) in a category $\mathscr{C}$ is a monoid in the strict monoidal category $\mathrm{End}(\mathscr{C})$ of endomorphisms of $\mathscr{C}$ (the tensor product is composition). That is, a monad is a triple (T, η, μ) consisting of a functor $T : \mathscr{C} \to \mathscr{C}$, and a pair of natural transformations $\eta : \mathbf{1} \to T$ and $\mu : T^2 = TT \to T$, such the diagrams

$$\begin{tikzcd} T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T\mathbf{1} \arrow[r, "T\eta"] \arrow[dr, "\mathrm{id}_T"'] & T^2 \arrow[d, "\mu"] & T\mathbf{1} \arrow[l, "\eta T"'] \arrow[dl, "\mathrm{id}_T"] \\ & T & \end{tikzcd}$$

(associativity) (identity)

commute. Morally, the first diagram attests to the fact that μ is "associative", independent of the order in which we collapse nested combinations, while the second diagram states that η is a double-sided unit of the collapsing morphism μ in an appropriate sense.

In analogy with the case of algebras of an endofunctor above, we would like a notion of algebras over a monad—we just demand that an algebra over a monad is an algebra over the underlying endofunctor, possessing a generic compatibility between the monad structure and algebra structure:

[2]With a few towards initial algebras of an endofunctor see for example [2], and with a view towards free algebras see [9].

Definition 2.2. If (T,η,μ) is a monad in a category $\mathscr{C}$, an *algebra over the monad* T is an algebra (C,f) over the endofunctor T that is compatible with the monadic structure, in that the diagrams (we will drop the parentheses from $T(C)$ from now on)

$$\begin{tikzcd} T^2C \arrow[r, "Tf"] \arrow[d, "\mu"'] & TC \arrow[d, "f"] \\ TC \arrow[r, "f"] & C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "\mathrm{id}_C"'] & TC \arrow[d, "f"] \\ & C \end{tikzcd}$$

(associativity) (identity)

commute. We will shortly see how the associativity and identity axioms for a T-algebra are concretely related to the corresponding axioms for the *signature* (parent) monad T. A morphism of algebras over a monad is then just a morphism of the underlying endofunctors.

We denote the category of all algebras over a monad T in a category $\mathscr{C}$ by $\mathscr{C}^T$. This category is also called the *Eilenberg-Moore category of* T. We may also call such a pair (C,f) a (T,η,μ)*-algebra*, and if the algebras over the endofunctor T are not also being considered in their own right, we often suppress the latter elements of the tuple and simply write T-algebra.

Henceforth unless otherwise specified, by "algebra" we will mean an algebra over a monad. Given our definition of algebras (C,f) over $T:\mathscr{C}\to\mathscr{C}$ as simply an object of $\mathscr{C}$ equipped with some additional structure, we immediately have access to a forgetful functor $U^T:\mathscr{C}^T\to\mathscr{C}$ which just forgets f. We will begin our investigation by studying some properties of this forgetful functor. As a warm-up, we first observe a lemma which will come in handy later on.

Lemma 2.3. *Let T be a monad in $\mathscr{C}$. Then the forgetful functor $U^T:\mathscr{C}^T\to\mathscr{C}$ reflects isomorphisms, in the sense that if we ever have that U^Tk is an isomorphism for a morphism k in $\mathscr{C}^T$, then k was already an isomorphism.*

Proof. Suppose that $k:(C,f)\to(C',f')$ is a morphism of T-algebras, and that $U^Tk:C\to C'$ is an isomorphism. To show that U^T reflects isomorphisms, it suffices to show that the inverse morphism $k^{-1}=(U^Tk)^{-1}:C'\to C$ is a morphism of T-algebras. By definition this amounts to checking that $f\circ Tk^{-1}=k^{-1}\circ f'$, but by the invertibility of k this equation is simply a rearrangement of the statement that k is a morphism of T-algebras. □

Given a forgetful functor, the natural question is "does this functor have an adjoint F^T?". The following proposition gives a positive answer! Indeed, taking the definition of monads and algebras over them at face value, the existence of F^T will seem a miraculous combination of the conditions we have placed on the structures involved. In contrast, there is no reason why we should expect an analogous result for algebras over endofunctors. The proof is a excellent example of "following ones nose".

Proposition 2.4. *The forgetful functor $U^T:\mathscr{C}^T\to\mathscr{C}$ has a left adjoint $F^T:\mathscr{C}\to\mathscr{C}^T$, with a unit $\eta^T:\mathbf{1}\to U^TF^T$, and a counit $\varepsilon^T:F^TU^T\to\mathbf{1}$.*

Proof. Given an object $C\in\mathscr{C}$, we must pair another object $C'\in\mathscr{C}$ with a morphism $TC'\to C'$ in such a way that we obtain at T-algebra. The only morphism which "unwraps a layer of T" which we have at our disposal is the component $\mu_C:T^2C\to TC$. Therefore, the only reasonable way to proceed is to define $F^T(C)=(TC,\mu_C)$. The associativity axiom of a structure map is then immediately satisfied by the corresponding one for the parent monad T, and similarly for the case of the identity axiom (in the latter, only one half[3] of the diagram for the monad T is required). The natural way to define F^T on a morphism $k:C\to C'$ in $\mathscr{C}$ in order that we obtain a morphism $TC\to TC'$ is by simply returning the result of acting on k with T, and so we must check that the diagram

$$\begin{tikzcd} T^2C \arrow[r, "T^2k"] \arrow[d, "\mu_C"'] & T^2C' \arrow[d, "\mu_{C'}"] \\ TC \arrow[r, "Tk"] & TC' \end{tikzcd}$$

commutes in each case. Fortunately, this is precisely the condition that μ is natural!

Noting that our definition of F^T gives that $T=U^TF^T$, we immediately recognise $\eta:\mathbf{1}\to T=U^TF^T$ as a candidate for the unit of an adjunction. We similarly seek a natural transformation $\varepsilon^T:F^TU^T\to\mathbf{1}$. For each

[3] One might find this asymmetry unsatisfying, but this concern will soon be beautifully rectified!

T-algebra (C, f), we require a morphism $(TC, \mu_C) \to C$, which is just an underlying morphism (satisfying a condition) $TC \to C$. The "only option" is thus f itself, and hence we take $\varepsilon_{(C,f)} = f$. That $\varepsilon_{(C,f)}$ is then a morphism of T-algebras follows immediately from the associativity axiom for the monad T. Naturality of ε is the requirement that for every morphism $k : (C, f) \to (C', f')$ of T-algebras the square

$$\begin{tikzcd} F^TU^T(C,f) \arrow[r, "\varepsilon_{(C,f)}"] \arrow[d, "FU^Tk"] & (C,f) \arrow[d, "k"] \\ F^TU^T(C',f') \arrow[r, "\varepsilon_{(C',f')}"] & (C',f') \end{tikzcd} \quad \text{i.e.} \quad \begin{tikzcd} (TC,\mu_C) \arrow[r, "f"] \arrow[d, "Tk"] & (C,f) \arrow[d, "k"] \\ (TC',\mu_{C'}) \arrow[r, "f'"] & (C',f') \end{tikzcd}$$

commutes. This is precisely the statement that every such k is a morphism of T-algebras, which is true by hypothesis.

The triangle identities are then

$$\begin{tikzcd} TC \arrow[r, "T\eta_C"] \arrow[dr, "\mathrm{id}_{TC}"'] & T^2C \arrow[d, "\mu_C"] \\ & TC \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "\mathrm{id}_C"'] & TC \arrow[d, "f"] \\ & C \end{tikzcd} .$$

The former is just the half of the identity axiom of the monad T which we didn't use above, and the latter is the identity axiom for T-algebras. This completes the proof. (In future we will generally become more terse when verifying auxiliary conditions, such as the triangle identities for adjunctions for example.) □

Intuitively, a left adjoint to a forgetful functor should be a "free functor", and so this motivates the name *free T-algebra functor* for F^T above. Moreover, we call $F^T(C) = (C, \eta_C)$ the *free T-algebra on C* in $\mathscr{C}$ (these are honest free objects satisfying a universal property in $\mathscr{C}^T$).

By the previous proposition, we can associate an adjunction $(F^T, U^T, \eta, \varepsilon)$ to every monad (T, η, μ) in a category $\mathscr{C}$. In fact, all of the data presented in this adjunction can be used to immediately recover the monad; we have $U^TF^T = T$, and η is present in both tuples. Furthermore, from the definition of ε above we see that $\varepsilon_{F^T(C)} = \mu_C$ as a morphism in $\mathscr{C}^T$ for every $C \in \mathscr{C}$. Hence we have the equation[4]

$$(T, \eta, \mu) = (U^TF^T, \eta, U^T\varepsilon^TF^T).$$

Of course, one immediately wonders whether we can obtain monads from general adjunctions; in fact, attempting the same construction as that executed above leads directly to the following proposition.

Proposition 2.5. *Every adjunction $(F, G, \eta, \varepsilon) : \mathscr{C} \to \mathscr{D}$ gives rise to a monad $(GF, \eta, G\varepsilon F)$.*

Proof. We only need to check the monad associativity and identity axioms. It is a fact, by the well-definedness of horizontal composition of natural transformations, that for arbitrary adjunctions and $D \in \mathscr{D}$ the diagram

$$\begin{tikzcd} FGFGD \arrow[r, "FG\varepsilon_D"] \arrow[d, "\varepsilon_{FGD}"] & FGD \arrow[d, "\varepsilon_D"] \\ FGD \arrow[r, "\varepsilon_D"] & D \end{tikzcd} \tag{1}$$

commutes. Applying G to this diagram, and taking $D = FC$ for $C \in \mathscr{C}$, we obtain the associativity axiom. Each triangle of the identity axiom is obtained by applying F or G to one of the triangle identities for the adjunction. □

We now return to the example of the previous section, reformulating algebraic groups as T-algebras over some monad T—in fact, our present developments permit this to be done in a canonical way. We begin with *only* the data of the forgetful functor $U : \mathsf{Groups} \to \mathsf{Sets}$. Up to isomorphism, U completely determines the free group functor F and indeed the entire adjunction $(F, U, \eta, \varepsilon) : \mathsf{Sets} \to \mathsf{Groups}$. Then by the previous proposition, we get a monad $(T, \eta, \mu) = (FU, \eta, U\varepsilon F)$. We will briefly sketch the properties of a T-algebra $(S, h) \in \mathsf{Sets}^T$:

[4] For convenience, we have employed the notation εF^T for the natural transformation defined in components by $(\varepsilon F^T)_C = \varepsilon_{F^T(C)}$.

- The morphism $h : TS \to S$, which we will suggestively call the *multiplication*, is just a function from the underlying set of the free group on S to S which is subject to two diagram identities. In this way, the set TS is in a concrete sense the "container of all combinations of elements of S", despite the fact that T has been indirectly constructed.

- As elements of sets, we have $\eta_S(x) = \langle x\rangle \in TS$, with η_S sending objects of S to the word in one element on them; this is morally what we specified that η_S should do. The identity axiom for the algebra is then just the statement that $h(\langle x\rangle) = x$ for every $x \in X$ (and so specifies h on the 1-element words).

- By inspection, the morphism $\mu_C : T^2S \to TS$ sends a word in words on elements of S to a single word formed by their concatenation (the inverse of a word is sent to the inverse of its constituent atoms taken in reverse order). The associativity axiom for the algebra is then the statement that $h(\langle xh(\langle yz\rangle)\rangle) = h(\langle xyz\rangle) = h(\langle h(\langle xy\rangle)z\rangle)$ (and similarly for longer words). If h is interpreted as a rule for multiplying the elements of the word passed to it, this is exactly the statement of associativity of the multiplication!

Thus, the monad algebra axioms conspire to require that we exactly have the data of a group multiplication (details such as the identity are compactly encoded as $h(\langle\rangle)$ using the empty word, and inverses are similar).

In this way we have encoded the definition of a group as exactly the monad T. If one wished, they could now completely forget[5] the definition of a group in terms of an underlying set, a binary operation, an inverse operation, identity, and some relations, and instead decide that a group is just any algebra over the monad T! Even more remarkably, we recovered all of this information from the forgetful functor $U : \mathsf{Groups} \to \mathsf{Sets}$ alone!

Of course, giving a monad (T,η,μ) (or in this case U) is considerably more structure than just giving its underlying endofunctor T. Indeed, forgetting all but the endofunctor in our example of the group signature monad above leaves only a functor which takes a set and gives the free group on that set as a set—and there are many more algebras over this endofunctor than there are groups! Thus one might be concerned that the monad axioms might excessively restrict the kinds of structures they can represent. However, by the end of our development we will establish a partial correspondence[6] between algebras over endofunctors and algebras over monads which greatly restores this confidence.

Given the present example, we must stress that the motivation given here for conceiving of the idea of monads, though natural and having elementary roots, is by no means the only perspective. As we noted above monads are also just particular monoid objects in an endofunctor category, or put differently again (and much less precisely), categorifications of idempotents ([16]). Indeed, historically[7] the dual notion of *comonads* came before the monads themselves, with the former appearing in the study of homological algebra of Godement [7].

3 Comparison and monadicity

Given an adjunction (F,G,η,ε), we have just seen that we can obtain an associated monad $(T,\eta,\mu) = (GF,\eta,G\varepsilon F)$. We can then construct a second adjunction $(F^T,U^T,\eta,\varepsilon^T)$, now associated to this monad. Although we can recover they same monad from both adjunctions, they need not be equivalent[8].

Going forward, a central consideration will be to understand how the original adjunction compares to the one derived from the Eilenberg-Moore construction. If an adjunction is "very close" to the one derived from its monad, we can consider as information that the original adjunction was "close" to an algebraic one. In order to make this precise, we introduce the following definition.

Definition 3.1. Let $(F : \mathscr{C} \to \mathscr{D}, G, \eta, \varepsilon)$ and $(F' : \mathscr{C} \to \mathscr{D}', G', \eta', \varepsilon')$ be adjunctions. A functor $X : \mathscr{D} \to \mathscr{D}'$ is a *comparison* of the former to the latter adjunction if the triangles

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[dr, "F'"'] & \mathscr{D} \arrow[d, "X"] \\ & \mathscr{D}' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{C} & \mathscr{D} \arrow[l, "G"'] \arrow[d, "X"] \\ & \mathscr{D}' \arrow[ul, "G'"] \end{tikzcd}$$

commute up to natural isomorphism. A comparison is *strict* if these diagrams are exact equalities.

[5]If the reader is dismayed by the prospect of having to retain knowledge of the domain of discourse Sets (or considers this overly restrictive), the should pursue the study of Lawvere theory ([14] is a famous text on the subject).

[6]This is Theorem 7.2.

[7]A brief history of the development of monads is given in [11].

[8]However, interpreted in a suitable 2-categorical sense, these are (incredibly) adjoint notions! See [15].

A comparison of the above type is of little use if it does not exist. The following proposition establishes that we always have a comparison between an arbitrary adjunction and the corresponding adjunction derived from the Eilenberg-Moore construction.

Proposition 3.2. *Let $(F, G, \mu, \varepsilon) : \mathscr{C} \to \mathscr{D}$ be an adjunction, and let $(T, \mu, G\varepsilon F)$ be the monad constructed therefrom. Furthermore, let $(F^T, U^T, \mu, \varepsilon^T)$ be the adjunction obtained from T via the Eilenberg-Moore construction. Then there exists a unique strict comparison $X : \mathscr{D} \to \mathscr{C}^T$.*

Proof. By direct calculation, we will propose a definition for the value of the strict comparison functor $X : \mathscr{D} \to \mathscr{C}^T$ on objects and morphisms based on its stated properties. We will then verify uniqueness.

To show existence, we must construct a functor $X : \mathscr{D} \to \mathscr{C}^T$ such that $U^T X = G$, and $XF = F^T$. On objects, this X acts as $X(D) = (x_D, h_D)$ for $x_D \in \mathscr{C}$ an object and $h_D : Tx_D \to x_D$ a morphism. In this language, the first condition $U^T X = G$ says that on objects $D \in \mathscr{D}$ we have $x_D = GD$. The second condition requires that $(x_{FC}, h_{FC}) = (TC, \mu_C) = (G(FC), G(\varepsilon_{FC}))$ for every $C \in \mathscr{C}$. We must define X on all of $\mathscr{D}$, and so we make the definition $h_D = G\varepsilon_D$ for all $D \in \mathscr{D}$. The identity axiom for T-algebras is just one of the triangle identities (and thus holds). Meanwhile, the commutativity of the associativity square is a general fact about adjunctions which follows again from (1). Finally, the comparison conditions give the definition $U^T X(f : D \to D') = G(f)$, and naturality of ε ensures that this is always a morphism of T-algebras. It is then immediate that these definitions for X actually define a functor.

It remains to ensure that our "choice" of h_D above is actually determined by the fact that X is a comparison. To do this, we could directly calculate using the definition of a comparison and the triangle identities. However, this is just a specific instance of a general fact[9] about morphisms of adjunctions (in this case we have a pair $X : \mathscr{D} \to \mathscr{C}^T$ and $\mathbf{1} : \mathscr{C} \to \mathscr{C}$, and they together define a morphism of adjunctions because $F \dashv G$ and $F^T \dashv U^T$ have an identical unit); we always have $\varepsilon^T_{XD} = X(\varepsilon_D)$. Unwrapping definitions, this equation becomes $h_D = G\varepsilon_D$, and this completes the proof. □

The encroachment of the evil notion of "strict" comparison functors may appear at first quite concerning. However, the existence of a canonical and even unique strict comparison functor will serve to greatly simplify some of the arguments which follow, and regardless it is straightforward to modify all of our strict arguments into ones that hold for any functor isomorphic to X. Indeed, it easily seen that relaxing the strictness condition in the above proof gives that comparisons of the above type are unique up to isomorphism.

Given the previous proposition, it makes sense to refer to *the* comparison functor associated to an arbitrary adjunction $(F, G, \eta, \varepsilon)$ (we implicitly compare the adjunction to the $(F^T, U^T, \eta, \varepsilon^T)$ construction). If this comparison functor is actually an equivalence of categories, then the functor G is called *monadic* (i.e. G is monadic if G has a left adjoint, and the comparison corresponding to this adjunction is an equivalence).

There is great value in knowing that a given functor under investigation is monadic, and our main goal from now on will be to determine this in general. For one thing, it will follow as a consequence of our development that the forgetful functor from the category of algebraic structures (made precise in a suitable sense) is monadic, a fact which we can also manually check with relative ease in familiar categories such as Groups, Rings, and Vec. If a functor U is monadic, it is therefore "algebraic" in the concrete sense that it behaves the same way as a forgetful functor from an algebraic structure would. However, algebra is by no means the only context to which monadicity is relevant; monadic functors also play a critical role in the theory of *monadic descent* (as suggested by the name), generalising Grothendieck's ideas of descent in sheaf theory ([8, 5]).

4 Assorted cutlery: our new best friends (or, Forks)

In this section we will study the properties of the comparison functor $X : \mathscr{D} \to \mathscr{C}^T$ defined above. In doing so, we will begin to assemble the machinery necessary to decide whether X is an equivalence.

If only motivated by the intention to remove a redundant arrow, we first observe that the associativity axiom for a T-algebra (C, f) is exactly encoded in the commutative diagram

$$T^2C \underset{Tf}{\overset{\mu_C}{\rightrightarrows}} TC \xrightarrow{\quad f \quad} C. \tag{2}$$

Pursuing our new notation further, we see that the identity axiom gives a right inverse η_C of f, while one of the identity laws for the entire monad T provides that η_{TC} is a right inverse of μ_C. This data might be encoded in the

[9] See §IV.7 of [11].

diagram

$$\begin{tikzcd} T^2C \arrow[r, shift left, "\mu_C"] \arrow[r, shift right, "Tf"'] & TC \arrow[l, bend right, "\eta_{TC}"'] \arrow[r, "f"] & C. \arrow[l, bend right, "\eta_C"'] \end{tikzcd} \tag{3}$$

There is no reason to expect that η_{TC} is also a left inverse of Tf. However, naturality of η does give that the compositions $Tf \circ \eta_{TC}$ and $\eta_C \circ f$ are equal!

We have just discovered an extremely general construction. Indeed, the commuting diagram of the type of (2) is known as a *fork*. Even more remarkably, (noncommutative) diagrams of the type of (3) along with the relations we obtained along with it are known as *splittings* of the above fork. We immediately introduce the formal definitions:

Definition 4.1. A *fork* (or sometimes, *cofork*) is a triple of three morphisms h, h', k in a category $\mathscr{C}$, specified by the structure and commutativity of the diagram

$$\begin{tikzcd} C \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & C' \arrow[r, "k"] & C'' \end{tikzcd} . \tag{4}$$

We say that k defines a fork on the parallel pair (h, h').

A *splitting* of the fork specified above is a pair of "section" morphisms $s : C'' \to C'$ and $r : C' \to C$ as in the (noncommutative) diagram

$$\begin{tikzcd} C \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & C' \arrow[l, bend right, "r"'] \arrow[r, "k"] & C'' \arrow[l, bend right, "s"'] \end{tikzcd} , \tag{5}$$

such that we have the equalities $k \circ s = \mathrm{id}_{C''}$, $h \circ r = \mathrm{id}_{C'}$, and $h' \circ r = s \circ k$. The name is inspired by a splitting of short exact sequences $C \xrightarrow{h} C' \xrightarrow{k} C''$ in homological algebra, with an extra condition added to give compatibility between h' and the splitting morphisms. A fork with a splitting is said to be *split*.

We now turn to the question of characterising the diagram (3), and in particular, how the morphism f can be substituted for another morphism while preserving the fact that we have a split fork. After making the following definition, which specifies how forks can be universal among all those on the same parallel pair, we will see that in fact we have very little choice at all.

Definition 4.2. A morphism k in a fork (4) is a *coequaliser* of the parallel pair (h, h') if it is universal in the following sense: For every other morphism $l : C' \to A$ defining a fork on (h, h'), there exists a unique morphism $\hat{l} : C'' \to A$ such that the diagram

$$\begin{tikzcd} C \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & C' \arrow[r, "k"] \arrow[dr, "l"] & C'' \arrow[d, "\hat{l}"] \\ & & A \end{tikzcd}$$

commutes. If the morphism $k : C' \to C''$ in question is clear from context, we sometimes call the object C'' the coequaliser (which is justified to the extent that it is unique up to unique isomorphism).

In fact, the following lemma shows that the split fork of (3) identified above specifies a coequaliser for very general reasons.

Lemma 4.3. *If a fork* (4) *is split, then k is a coequaliser of the pair* (h, h').

Proof. For any morphism $l : C' \to A$ defining another fork on (h, h'), we have a (noncommutative) diagram

$$\begin{tikzcd} C \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & C' \arrow[l, bend right, "r"'] \arrow[r, "k"] \arrow[dr, "l"] & C'' \arrow[l, bend right, "s"'] \\ & & A \end{tikzcd}$$

for s and r together a splitting for the upper fork. Then, such morphisms l are in bijection with morphisms $m : C'' \to A$ by the maps (we just use the morphisms k and s fix the source and target) $l \mapsto l \circ s$ and $m \mapsto m \circ k$. Indeed, we can directly compute $(l \circ s) \circ k = l \circ (h' \circ r) = (l \circ h) \circ r = l$ and $(m \circ k) \circ s = m \circ (k \circ s) = m$. This is exactly the universal property stating that k is a coequaliser, as desired. ☐

Thus the parallel pair $(\mu_C, Tf) = (U\varepsilon_{FC}, UFf)$ is always part of a split fork. We call the pair (ε_{FC}, Ff) is *U-split*, because it is split under the image of U.

Our overarching goal is to determine when the comparison functor $X : \mathscr{D} \to \mathscr{C}^T$ is an equivalence of categories. As every equivalence can be refined into an adjoint equivalence, in this case X would certainly have to have a left-adjoint $Y : \mathscr{C}^T \to \mathscr{D}$. The preceding discussion, wishful thinking, and the fact that universality of coequalisers hints at being able to select objects and morphisms in a functorial way, then leads us to consider the following proposition.

Proposition 4.4. *Suppose that $\mathscr{D}$ has coequalisers of U-split pairs. Then the comparison functor $X : \mathscr{D} \to \mathscr{C}^T$ has a left adjoint $Y : \mathscr{C}^T \to \mathscr{D}$.*

Proof. For each $(C, f) \in \mathscr{C}^T$, by the discussion above we have a split fork (we have simply expanded (3) using $T = UF$)

$$\begin{tikzcd} UFUFC \arrow[r, shift left, "\mu_C"] \arrow[r, shift right, "UFf"'] & UFC \arrow[l, bend right, "\eta_{UFC}"'] \arrow[r, "f"] & C. \arrow[l, bend right, "\eta_C"'] \end{tikzcd} \tag{6}$$

Now, we have $\mu_C = U\varepsilon_{FC}$, and therefore the pair (ε_{FC}, Ff) is U-split. Therefore, by hypothesis there exists a coequaliser morphism $q_{(C,f)} : FC \to Y(C, f)$ in $\mathscr{D}$ for this parallel pair (and we define the functor Y on objects in this way). This fits into a coequaliser fork diagram (using the fact that $\mu_C = U(\varepsilon_{FC})$)

$$\begin{tikzcd} FUFC \arrow[r, shift left, "\varepsilon_{FC}"] \arrow[r, shift right, "Ff"'] & FC \arrow[r, "q_{(C,f)}"] & Y(C, f). \end{tikzcd} \tag{7}$$

We now need to define Y on morphisms. To this end, let $\phi : (C, f) \to (C', f')$ be arbitrary. In order to identify what the image of ϕ under Y must be, we juxtapose the natural diagrams we have involving $Y(C, f')$ and $Y(C', f')$ (these diagrams constitute our only information regarding the definitions of $Y(C, f')$ and $Y(C', f')$!):

$$\begin{tikzcd} FUFC \arrow[r, shift left, "\varepsilon_{FC}"] \arrow[r, shift right, "Ff"'] \arrow[d, "FUF\phi"] & FC \arrow[r, "q_{(C,f)}"] \arrow[d, "F\phi"] & Y(C, f) \\ FUFC' \arrow[r, shift left, "\varepsilon_{FC'}"] \arrow[r, shift right, "Ff'"'] & FC' \arrow[r, "q_{(C',f')}"] & Y(C', f') \end{tikzcd}. \tag{8}$$

Our goal is to factor $q_{(C',f')} \circ F\phi$ through $q_{(C,f)}$ by the fact that the latter is a coequaliser, and therefore obtain a morphism $Y\phi : Y(C, f) \to Y(C', f')$. We are in luck; each of the two superimposed squares above commute respectively by the naturality of ε, and by simply applying F to the diagram stating that ϕ is a morphism of T-algebras. Therefore the composition $q_{(C',f')} \circ F\phi$ is a fork on the parallel pair (ε_{FC}, Ff), from which it follows immediately from the universal property of coequalisers that there is a unique morphism $Y\phi : Y(C, f) \to Y(C', f')$ so that $q_{(C,f)} \circ Y\phi = q_{(C',f')}$. We define Y on morphisms in this way, and it follows quickly from the universal property of coequalisers that Y is a functor.

It remains to provide a unit and counit for the claimed adjunction. First, to construct the unit $\sigma : \mathbf{1} \to XY$, for each $(C, f) \in \mathscr{C}^T$ we must provide a morphism $\sigma_{(C,f)} : (C, f) \to XY(C, f) = (UY(C, f), U\varepsilon_{Y(C,f)})$. This is just an underlying morphism $C \to UY(C, f)$ in $\mathscr{C}$ which obeys the T-algebra conditions. Our definition of the functor Y is essentially provided by diagrams of the form (7), and we have little choice in how to obtain a morphism with target $UY(C, f)$ therefrom; we must apply the functor U to the diagram! This gives

$$\begin{tikzcd} T^2C \arrow[r, shift left, "\mu_C"] \arrow[r, shift right, "Tf"'] & TC \arrow[r, "Uq_{(C,f)}"] & UY(C, f), \end{tikzcd}$$

after using the definitions of T and μ_C to tidy up the labels somewhat. This diagram strongly resembles that of the split fork (6), which we used to bootstrap the entire construction of Y. Indeed, because of the presence of a splitting, the coequaliser morphism $f : TC \to C$ fits into the above diagram. If the image of the coequaliser $q_{(C,f)}$ under U remained a coequaliser, then we would be able to make the strong conclusion that $C \cong UY(C, f)$ in a unique way. Unfortunately, this need not be the case. Instead, the coequaliser universal property gives a unique morphism $\sigma_{(C,f)} : C \to UY(C, f)$ fitting into the commutative diagram

$$\begin{tikzcd} T^2C \arrow[r, shift left, "\mu_C"] \arrow[r, shift right, "Tf"'] & TC \arrow[r, "Uq_{(C,f)}"] \arrow[dr, "f"'] & UY(C, f) \\ & & C, \arrow[u, "\sigma_{(C,f)}"'] \end{tikzcd}$$

and this is precisely how we will define σ. In doing this for each $(C, f) \in \mathscr{C}^T$ we must check that we obtain a morphism in $\mathscr{C}^T$, and not just $\mathscr{C}$. However, the necessary commuting square follows after expressing the "upper path" using the pair of double forks in (8) used to define Y on morphisms, and the lower path as the coequaliser $q_{(C,f)}$ (by definition). The universal property of coequalisers then forces the diagram to commute. By using the trick above of obtaining two forks and a pair of commuting superimposed squares therefrom, the universal property of coequalisers also gives immediately that the components of σ are natural.

In order to define the counit $\tau : YX \to \mathbf{1}$, for each $D \in \mathscr{D}$ we must give a morphism $\tau_D : YXD = Y(UD, U\varepsilon_D) \to D$ (and these morphisms must be compatible). We proceed in a very similar way to that above; first, the diagram in which the source object of this morphism naturally appears (indeed, the diagram defines $Y(UD, U\varepsilon_D)$ up to isomorphism) is

$$\begin{tikzcd} FUFUD \arrow[r, shift left, "\varepsilon_{FUD}"] \arrow[r, shift right, "FU\varepsilon_D"'] & FUD \arrow[r, "q_{(UD,U\varepsilon_D)}"] \arrow[dr, "\varepsilon_D"] & Y(UD, U\varepsilon_D) \\ & & D, \end{tikzcd} \tag{9}$$

where we have inserted the morphism ε_D for our imminent convenience. By the fact that $q_{(UD,U\varepsilon_D)}$ is a coequaliser, we immediately get a unique factoring of $\varepsilon_D : FUD \to D$ as $\tau_D \circ q_{(UD,U\varepsilon_D)} = \varepsilon_D$, and so this provides our desired morphism τ_D. Once again, by using the "superimposed squares built from forks trick", we easily verify that these components define a natural transformation τ. Finally, the triangle identities are also proved by this trick; for the one involving Y, we now focus on the pair of forks used in the definition of Y on morphisms, while for the triangle involving X, we juxtapose the forks defining the components of σ and τ. $\square$

5 On the unreasonable effectiveness of coequalisers

It would be dishonest at this point to conclude the proof of Proposition 4.4 without a minor digression which helps to explain the success we had with our method of constructing the left adjoint. First note that, despite the fact that the hypothesis of the proposition demands that "$\mathscr{D}$ has coequalisers of U-split pairs", we only applied this property to diagrams of the form (e.g. in (7))

$$\begin{tikzcd} FUFC \arrow[r, shift left, "\varepsilon_{FC}"] \arrow[r, shift right, "Ff"'] & FC \arrow[r, dotted] & {} \end{tikzcd} .$$

There, we asked for a coequaliser morphism to fill the place of the dotted arrow. We could therefore ask for the existence of coequalisers of only these diagrams, and still obtain our desired left adjoint.

Furthermore, when defining the counit τ of the adjunction $Y \dashv X$ for example, we extended this diagram in the case $C = UD$ by filling in the dotted arrow with ε_D. In doing this, we have actually arrived at yet another very general construction which is possible for any adjunction $(F, G, \eta, \varepsilon) : \mathscr{C} \to \mathscr{D}$. Namely, in (9) we drew for each $D \in \mathscr{D}$ a diagram

$$\begin{tikzcd} FGFGD \arrow[r, shift left, "\varepsilon_{FGD}"] \arrow[r, shift right, "FG\varepsilon_D"'] & FGD \arrow[r, "\varepsilon_D"] & D \end{tikzcd} . \tag{10}$$

In fact, this is a fork by the naturality diagram of ε associated to the morphism $\varepsilon_D : FGD \to D$! It is famously called the *canonical resolution* or *canonical presentation* of D for the adjunction, for in the case of the free-forgetful adjunction Groups $\to$ Sets it presents arbitrary groups G as the quotient of the free group on the elements of G (the quotient is expressed[10] as a coequaliser).

Specialising to the case $(F, G, \eta, \varepsilon) = (F^T, U^T, \eta, \varepsilon)$ for T a monad, the canonical resolution diagram for $(C, f) \in \mathscr{C}^T$ is

$$\begin{tikzcd} F^TU^TF^TC \arrow[r, shift left, "\varepsilon_{F^TC}"] \arrow[r, shift right, "F^TU^T\varepsilon_{(C,f)}"'] & F^TC \arrow[r, "\varepsilon_{(C,f)}"] & (C, f) \end{tikzcd} ,$$

which upon evaluating the definitions of the functor involved and relabelling is just

$$\begin{tikzcd} (T^2C, \mu_{TC}) \arrow[r, shift left, "\mu_C"] \arrow[r, shift right, "Tf"'] & (TC, \mu_C) \arrow[r, "f"] & (C, f) \end{tikzcd} . \tag{11}$$

Applying U^T to the diagram, we obtain the fork which we have already[11] seen, in (2); it originally catalysed our investigation of coequalisers and in fact splits. Because the splitting morphisms η_C and η_{TC} are not morphisms

[10] In particular, one of the parallel morphisms takes a word in words in group elements and evaluates the innermost words using the group multiplication, while the other simplifies words in words down to a single word in the generic way.

[11] Indeed, there truly is a conspiracy afoot!

of T-algebras (in general), there is no reason why (11) should split. Regardless, the existence of a coequaliser after applying U^T may be used to bootstrap obtaining one for the original diagram. This is the content of the following proposition, where we simply manually check that we can transport a coequaliser from one category to the other. It is a good example of the style of argument we will use to prove facts specifically about $\mathscr{C}^T$ and the functor $U : \mathscr{C}^T \to \mathscr{C}$; this will become of central importance in the next section.

Proposition 5.1. *For any $(C, f), (C', f') \in \mathscr{C}^T$ and $k : (TC, \mu_C) \to (C', f')$ making the diagram*

$$\begin{tikzcd}
(T^2C, \mu_{TC}) \arrow[r, shift left, "\mu_C"] \arrow[r, shift right, "Tf"'] & (TC, \mu_C) \arrow[r, "f"] \arrow[dr, "k"'] & (C, f) \arrow[d, dotted, "\hat{k}"] \\
& & (C', f')
\end{tikzcd} \tag{12}$$

commute (excluding the dotted arrow), there exists a unique $\hat{k} : (C, f) \to (C', f')$ such that $\hat{k} \circ f = k$.

Proof. We only briefly sketch the proof, because a more general version is included in the content of Proposition 6.3 of the next section. Using the fact that the top row of (12) splits under U^T we obtain a factoring $k = \hat{k} \circ f$ of underlying morphisms in $\mathscr{C}$, and $\hat{k}$ is our candidate to be upgraded to a morphism in $\mathscr{C}^T$. Combining the squares which state that f and k are morphisms of T-algebras, we once again use the fact that f is a coequaliser in $\mathscr{C}$ in order to conclude that $\hat{k}$ is a morphism of T-algebras as well. Uniqueness of $\hat{k}$ then follows from the fact that U^T is faithful. □

By definition, we have that τ is a natural isomorphism if and only if $\varepsilon_{(C,f)}$ appears as a coequaliser in the canonical resolution of the T-algebra (C, f). It is an elementary theorem of category theory that X is fully faithful if and only is the counit τ is invertible, and so τ on its own gives information about the comparison functor X. This property turns out to be of central importance in the theory of monadic descent ([8]), where in this case X is said to be of *descent type*. This provides another example of how canonical resolutions appear as a convenient language in which to speak about properties of the comparison functor X, and related ideas.

We conclude this section by establishing a partial[12] converse to Proposition 4.4, using the concept of canonical resolutions introduced above. The proposition requires the following small (but beautiful) observation.

Lemma 5.2. *Suppose that the (strict) comparison functor $X : \mathscr{D} \to \mathscr{C}^T$ above has a left adjoint $Y : \mathscr{C}^T \to \mathscr{D}$. Then $YF^T \cong F$.*

Proof. We have $Y \dashv X$, $F^T \dashv U^T$, and $F \dashv U$. Composing adjunctions, we thus have that $YF^T \dashv U^T X \cong U$. By the uniqueness of adjoints up to isomorphism, we thus have the desired isomorphism $YF^T \cong F$. □

What follows is the bedrock on which we found the remainder of our explorations. It ensures that the conditions investigated in Proposition 4.4 are far from unobtainable; indeed (under a suitable weakening of the hypotheses of that proposition), it establishes a converse and hence equivalence.

Proposition 5.3. *Suppose that the comparison functor $X : \mathscr{D} \to \mathscr{C}^T$ associated to an adjunction (F, U, μ, ε) has a left adjoint $Y : \mathscr{C}^T \to \mathscr{D}$. Then for every $(C, f) \in \mathscr{C}^T$ the parallel pair*

$$\begin{tikzcd}
FUFC \arrow[r, shift left, "\varepsilon^T_{FC}"] \arrow[r, shift right, "Ff"'] & FC
\end{tikzcd} \tag{13}$$

has a coequaliser in $\mathscr{D}$.

Proof. By Proposition 5.1 above, the canonical resolution of a T-algebra (C, f) defines a coequaliser in $\mathscr{C}^T$. Given the observation of Lemma 5.2, the idea is simply to apply Y to (11), yielding

$$\begin{tikzcd}
YF^T TC \arrow[r, shift left, "Y\varepsilon^T_{F^T C}"] \arrow[r, shift right, "YF^T f"'] & YF^T C \arrow[r, "Yf"] & Y(C, f)
\end{tikzcd} .$$

Critically, left adjoint functors preserve all colimits and in particular coequalisers, and therefore Yf is a coequaliser. By the natural isomorphism of Lemma 5.2, it suffices to show that $Y\varepsilon^T_{F^T C} = \varepsilon_{YF^T C}$—as then by pre- and post-composition with the components of the natural isomorphism we will obtain a coequaliser of a parallel pair matching (13). We could directly compute this using the homset bijections of the adjunctions, but this a general fact given that Y is the left adjoint to a comparison functor X (which is a morphism of adjoints, see §IV.7 of [11]). □

[12] The proposition is a full converse when the weaker hypothesis described above is used in the statement of Proposition 4.4.

6 Beck's monadicity theorem

As we noted above when constructing the unit of the adjunction $Y \dashv X$, the coequalisers $q_{(C,f)} : FC \to Y(C,f)$ for $(C,f) \in \mathscr{C}^T$ need not be sent to coequalisers under the functor $U : \mathscr{D} \to \mathscr{C}$. However, this would exactly be the case if U *preserves coequalisers of U-split pairs.* Then, the universality of coequalisers would imply that there is a unique isomorphism $(C,f) \to UY(C,f)$ in $\mathscr{C}^T$, and this isomorphism would exactly be the component $\sigma_{(C,f)}$. This would therefore make σ into a natural isomorphism. It is also timely to introduce the notion in some sense dual to preservation of coequalisers; we say U *reflects coequalisers of U-split pairs* if every fork which has a U-split image is itself already a coequaliser. The above observation then proves the first part of the following proposition.

Proposition 6.1. *Suppose that $\mathscr{D}$ has coequalisers of U-split pairs, so that the comparison functor $X : \mathscr{D} \to \mathscr{C}^T$ is part of an adjunction $(Y, X, \sigma, \tau) : \mathscr{C}^T \to \mathscr{D}$ by Proposition 4.4. Then*

- *if in addition U preserves coequalisers of U-split pairs, the unit σ is an isomorphism, and (independent of this point)*
- *if in addition U reflects coequalisers of U-split pairs, the counit τ is an isomorphism.*

Proof. The dual observation for the counit τ is only slightly more subtle; fix $D \in \mathscr{D}$. Applying U to diagram (9), we recognise the bottom row as the $(C,f) = XD$ special case of the split fork of (3). If then U reflects coequalisers of U-split pairs, we would be able to conclude that the morphism ε_D was a coequaliser in the original diagram. By definition $q_{(UD, U\varepsilon_D)}$ is a coequaliser in this original diagram and hence there is a unique isomorphism $YXD = Y(UD, U_{\varepsilon D}) \to D$ again by the universal property of coequalisers. This isomorphism is exactly the component τ_D, and therefore τ is a natural isomorphism in this case as well. □

By the remarks of the previous section, it follows that the second condition of Proposition 6.1 is a sufficient condition for U to be of descent type.

We are nearing our primary goal; together, Propositions 4.4 and 6.1 establish a collection of conditions that when satisfied imply that a given functor $U : \mathscr{D} \to \mathscr{C}$ with a left adjoint F is monadic. Indeed, we have proved the following proposition.

Proposition 6.2. *Let $U : \mathscr{D} \to \mathscr{C}$ be a functor. Then if*

- *the category $\mathscr{D}$ has coequalisers of U-split pairs, and*
- *the functor U has a left adjoint, and preserves and reflects coequalisers of U-split pairs,*

then U is monadic.

Proof. If the category $\mathscr{D}$ has coequalisers of U-split pairs then for $T = FU$, the comparison functor $X : \mathscr{D} \to \mathscr{C}^T$ of Proposition 3.2 fits into an adjunction $(Y, X, \sigma, \tau) : \mathscr{C}^T \to \mathscr{D}$. If U both preserves and reflects coequalisers of U-split pairs, then Proposition 6.1 provides that the unit $\sigma : \mathbf{1} \to XY$ and counit $\tau : YX \to \mathbf{1}$ are both actually natural isomorphisms. Hence, they together witness the fact that the functors $Y : \mathscr{C}^T \to \mathscr{D}$ and $X : \mathscr{D} \to \mathscr{C}^T$ together give an equivalence of categories. Therefore U is monadic. □

One is eager to determine exactly how close these conditions are to being sufficient, in the hope that the distance is not too great. We first turn to checking the obvious and most extreme case.

Considering the adjunction $(F^T, U^T, \eta, \varepsilon^T) : \mathscr{C} \to \mathscr{C}^T$ for a fixed monad T, the comparison $X : \mathscr{C}^T \to \mathscr{C}^T$ must be the identity by the uniqueness result of Proposition 3.2. The identity functor on $\mathscr{C}^T \to \mathscr{C}^T$ is certainly an equivalence of categories, and therefore if these conditions are to have any power whatsoever[13], the category $\mathscr{C}^T$ and functor U^T must satisfy them. This is the content of the following proposition.

Proposition 6.3. *For T a monad in $\mathscr{C}$, we have that*

- *the category $\mathscr{C}^T$ has coequalisers of U^T-split pairs, and*
- *the forgetful functor $U^T : \mathscr{C}^T \to \mathscr{C}$ both; (i) preserves, and (ii) reflects coequalisers of U^T-split pairs.*

[13] This claim is not entirely honest. For, in the previous section we have shown above that we have a left adjoint to X exactly when we have existence in $\mathscr{D}$ of coequalisers of the parallel pairs in the canonical resolutions of T-algebras. As $X : \mathscr{C}^T \to \mathscr{C}^T$ certainly has a left adjoint, these properties are then automatic for U^T. However, the content of Proposition 6.3 wholly supersedes the conditions of Proposition 5.3 in a nontrivial way, strengthening them to the existence in $\mathscr{C}^T$ of coequalisers of all U^T-split pairs.

Proof. We will actually prove a stronger version of the proposition statement, which implies the claimed result. Indeed, let the diagram

$$\begin{tikzcd} C \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & C' \arrow[r, "k"] & C'' \end{tikzcd}$$

define a coequaliser (and fork) in a category $\mathscr{C}$. This coequaliser is *absolute* if for any functor $J : \mathscr{C} \to \mathscr{A}$ to any other category $\mathscr{A}$ the image fork

$$\begin{tikzcd} JC \arrow[r, shift left, "Jh"] \arrow[r, shift right, "Jh'"'] & JC' \arrow[r, "Jk"] & JC'' \end{tikzcd}$$

is always still a coequaliser (in the new category $\mathscr{A}$). One might wonder at first how this is possible, but we have an obvious class of examples at our disposal; namely, as a split fork is certainly taken to a split fork by any functor, every split fork defines an absolute coequaliser. It remains to show that the hypotheses of the proposition hold with "split" replaced by "absolute" (in analogy with the "split" case, a parallel pair is U-absolute if its image under U is an absolute coequaliser).

To this end, suppose we have a parallel pair

$$\begin{tikzcd} (C, f) \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & (C', f') \end{tikzcd}$$

that in the image of U^T has an absolute coequaliser, as in the diagram (we have slightly abused notation by omitting U^T on morphisms)

$$\begin{tikzcd} C \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & C' \arrow[r, "q"] & C''. \end{tikzcd}$$

We desire a morphism $g : TC'' \to C''$ such that (C'', g) is a T-algebra, and to have g so chosen that $q : (C', f') \to (C'', g)$ is a morphism of T-algebras. Then if this q is actually a coequaliser in $\mathscr{C}^T$ we will have established point (ii). Point (i) is immediate by combining (ii) with the trivial fact that $U^T q = q$ (abusing notation).

Hence we need a morphism $g : TC'' \to C''$, and we already have a general method when dealing with coequaliser forks for obtaining one. Namely, we draw a diagram of two juxtaposed coequaliser forks

$$\begin{tikzcd} TC \arrow[r, shift left, "Th"] \arrow[r, shift right, "Th'"'] \arrow[d, "f"] & TC' \arrow[r, "Tq"] \arrow[d, "f'"] & TC'' \\ C \arrow[r, shift left, "h"] \arrow[r, shift right, "h'"'] & C' \arrow[r, "q"] & C'' \end{tikzcd}$$

such that the superimposed squares both commute (in this case they state that h and h' are each respectively T-algebra morphisms). Indeed, the lower-right morphism is a coequaliser by hypothesis, and the upper-right morphism is a coequaliser because q is an absolute one. As before, the composition $q \circ f'$ is therefore a coequaliser of the upper fork, and factors uniquely through Tq to give the desired morphism $g : TC \to T$. Because the inclusion of g into the above diagram preserves commutativity, this will also show that q is a morphism $(C', f') \to (C'', g)$ of T-algebras! (Technically, we have not yet established that (C'', g) is actually a T-algebra.)

To check that g satisfies the T-algebra structure map identity axiom, we take the right-hand square of the diagram above and redraw it with the morphisms g, $\eta_{C'}$, and $\eta_{C''}$ included so that the result is still commutative;

$$\begin{tikzcd} C' \arrow[r, "q"] \arrow[d, "\eta_{C'}"] \arrow[dd, bend right=60, "\mathrm{id}_{C'}"'] & C'' \arrow[d, "\eta_{C''}"'] \arrow[dd, bend left=60, "\mathrm{id}_{C''}"] \\ TC' \arrow[r, "Tq"] \arrow[d, "f'"] & TC'' \arrow[d, "g"'] \\ C' \arrow[r, "q"] & C'' \end{tikzcd}.$$

The diagram immediately gives $q = g \circ \eta_{C''} \circ q$. But q is a coequaliser and therefore the composite $g \circ \eta_{C''}$ is the unique such morphism $k : C'' \to C''$ such that $q = k \circ q$. The identity morphism certainly satisfies this property, and hence $g \circ \eta_{C''} = \mathrm{id}_{C''}$, as desired. In other words, we have used the fact that coequalisers are *epimorphisms*. The associativity axiom is proved in a completely analogous manner—we simply compare to the statement that the axiom holds for (C', f'), and then use the fact that q is a coequaliser.

It remains to show that q is actually a coequaliser in $\mathscr{C}^T$; to this end, let $p : (C, f) \to (A, h)$ be any morphism which has equal postcomposites with h and h'. Then p has underlying morphism $C' \to A$, and hence by the fact that q is a coequaliser in $\mathscr{C}$ there exists a unique $k : C'' \to A$ such that $k \circ q = p$. Diagrammatically, we have

$$\begin{tikzcd}
TC' \arrow[r, "Tq"] \arrow[d, "f'"] \arrow[rr, bend left, "Tp"] & TC'' \arrow[r, dotted, "Tk"] \arrow[d, "g"] & TA \arrow[d, "h"] \\
C' \arrow[r, "q"] \arrow[rr, bend right, "p"'] & C'' \arrow[r, dotted, "k"] & A
\end{tikzcd}$$

where the diagram is commutative without the dotted arrows, and additionally the top and bottom half-circles commute with the dotted arrow included. But this means that

$$(h \circ Tk) \circ Tq = h \circ Tp = (p \circ f') = k \circ (q \circ f') = (k \circ g) \circ Tq.$$

Because q is an absolute coequaliser, Tq is a coequaliser and hence epi. Thus $h \circ Tk = k \circ g$ and p is a morphism of T-algebras. Therefore q is a coequaliser in $\mathscr{C}^T$, and this completes the proof. □

In fact, one can extend Proposition 6.3 to the claim that U^T preserves and reflects all limits whatsoever ([4]). This is the abstract explanation as to why when one performs constructions such as taking products of algebraic structures, they do so on the underlying sets! Because U^T reflects isomorphisms by Lemma 2.3, we also obtain the familiar result that morphisms of algebraic structures that are bijective on the underlying sets are isomorphisms.

Now somewhat reassured by the statement of Proposition 6.3, we consider a monadic functor $U : \mathscr{D} \to \mathscr{C}$ defining a monad T. But in fact, because monadic functors provide an equivalence $\mathscr{D} \cong \mathscr{C}^T$, we have stumbled into the following main result!

Theorem 6.4 (Precise Monadicity Theorem, first version)**.** *Let $U : \mathscr{D} \to \mathscr{C}$ be a functor. Then U is monadic if and only if*

- *the category $\mathscr{D}$ has coequalisers of U-split pairs, and*
- *the functor U has a left adjoint, and preserves and reflects coequalisers of U-split pairs.*

Proof. As already mentioned, sufficiency is provided by Proposition 6.2. Now let $U : \mathscr{D} \to \mathscr{C}$ be monadic. By definition, the comparison $X : \mathscr{D} \to \mathscr{C}^T$ is an equivalence of categories. Then, by the additional identity $U^T X = U$, the hypotheses of Proposition 6.2 are equivalent to the specific case of taking $\mathscr{D} = \mathscr{C}^T$ and $U = U^T$. They are therefore verified by Proposition 6.3, and this completes the proof. □

It is striking that we have obtained a precise characterisation of monadic functors by primarily employing wishful thinking, and then recognising and analysing the structures with which we have been presented.

There are several remarks which now deserve to be made. One is that by proving the theorem via absolute coequalisers (instead of split ones), we actually possess a stronger half of the implication of Theorem 6.4; with U monadic implying the conclusions with "split" replaced with "absolute" (incidentally, the latter condition is often easier to check in practice). Furthermore, it was noted above when constructing a left adjoint to the comparison functor $X : \mathscr{D} \to \mathscr{C}^T$, that coequalisers for only a special class of U-split pairs were actually required. In fact, these pairs (which were of the form (ε_{FC}, Ff) for $(C, f) \in \mathscr{C}^T$) were all *reflexive*, in that they possessed a common section (i.e. right inverse) $F\eta_C$. The hypotheses of Theorem 6.4 are thus frequently weakened to only demand that $\mathscr{D}$ have reflexive U-split pairs. Also common is a further weakening by the application of the following lemma (taking P below to be the reflexive U-split pairs).

Lemma 6.5. *Let $U : \mathscr{D} \to \mathscr{C}$ be a functor which reflects all isomorphisms and preserves coequalisers of some set P of parallel pairs in $\mathscr{D}$. If $\mathscr{D}$ has coequalisers of elements of P, then U reflects coequalisers of elements of P.*

Proof. Suppose that we have a fork $D \overset{h,h'}{\rightrightarrows} D' \xrightarrow{k} D''$ in P for which Uk is a coequaliser. Then for $q : D' \to A$ a coequaliser in $\mathscr{D}$, there is a unique morphism $\phi : A \to D''$ such that $\phi \circ q = k$. We thus have a commuting diagram

$$\begin{tikzcd} UD \arrow[r, shift left, "Uh"] \arrow[r, shift right, "Uh'"'] & D' \arrow[r, "Uk"] \arrow[dr, "Uq"'] & D'' \\ & & A \arrow[u, "U\phi"'] \end{tikzcd}.$$

By the hypotheses of the lemma both Uk and Uq are now equalisers, and so $U\phi$ is an isomorphism. But U reflects isomorphisms, and therefore $\phi : A \to D''$ is an isomorphism. This proves that k is a coequaliser, as required. □

The reverse direction of the monadicity equivalence is then provided by Lemma 2.3, establishing the following.

Theorem 6.6 (Precise Monadicity Theorem, second version)**.** *Let $U : \mathscr{D} \to \mathscr{C}$ be a functor. Then U is monadic if and only if*

- *the category $\mathscr{D}$ has coequalisers of reflexive U-split pairs, and*
- *the functor U has a left adjoint, reflects isomorphisms, and preserves coequalisers of reflexive U-split pairs.*

Moreover, the theorem holds with "split" replaced by "absolute", and/or "reflexive" deleted.

Finally, this result (in its various forms) is also known as *Beck's theorem*, after Jonathan Mock Beck who proved it in the course of writing his PhD dissertation in 1964 ([3]). His original version supposed that the category $\mathscr{D}$ had all coequalisers and that a functor $U : \mathscr{D} \to \mathscr{C}$ preserved and reflected all of them, and then concluded that U was monadic.

7 Implications and applications

We now resolve the question which originally prompted our investigation, and make several brief observations of the applications and consequences of Theorem 6.6. First, observe that by excising the checks of the associativity and identity axioms of the structure maps for T-algebras appearing in the proof of Proposition 6.3, we obtain the following proposition and an immediate corollary.

Proposition 7.1. *For $T : \mathscr{C} \to \mathscr{C}$ any endofunctor, the forgetful functor $G : \mathbf{Alg}(T) \to \mathscr{C}$ preserves and reflects coequalisers of G-absolute (and hence G-split) pairs.*

Theorem 7.2. *Let $T : \mathscr{C} \to \mathscr{C}$ be an endofunctor, and $G : \mathbf{Alg}(T) \to \mathscr{C}$ the associated forgetful functor. If G has a left adjoint F, then the category $\mathbf{Alg}(T)$ is equivalent to $\mathscr{C}^{\overline{T}}$ with $\overline{T} = GF$ the monad obtained from the adjunction $F \dashv G$.*

Proof. By Proposition 7.1 $G : \mathbf{Alg}(T) \to \mathscr{C}$ satisfies the hypotheses of Theorem 6.6 and therefore is monadic. The comparison $X : \mathbf{Alg}(T) \to \mathscr{C}^{\overline{T}}$ is therefore an equivalence of categories. □

This (partially) resolves our long-outstanding question of which categories of algebras over endofunctors can be upgraded into algebras over a monad, in that the theorem reduces the question to one of whether a particular functor has a left adjoint. This is a much more general problem, and grants us access to the wide array of available adjoint functor theorems. For example, if $\mathscr{C}$ is complete we can obtain the conclusion of Theorem 6.6 by the General Adjoint Functor Theorem.

The monad $\overline{T}$ defined above is called the *algebraically-free monad generated by T*, in analogy with classical algebra. Specifically, the difference between an algebra over an endofunctor and an algebra over a monad is analogous to the difference between the action of a set $S \times X \to X$, and the action of a monoid $M \times X \to X$ on that set (the former being equivalent to the latter with M the free monoid generated by S).

The primary applications of Beck's theorem are roughly classified into a dichotomy; the first class consists of the far reaching consequences of Beck's theorem in modern algebraic geometry and sheaf theory. In particular, Beck's theorem plays a central role in descent theory, where monadicity allows categories of descent data to be identified with honest algebras over a particular monad ([8]). For example, the faithfully flat descent of Grothendieck's FGA and in SGA1 are specialisations of Beck's theorem ([12]). Beck's theorem has also found applications in the study of Tannakian categories, where Deligne (in [6]) has used the theorem to simplify the foundational development of the theory.

The other half of the dichotomy classifies results obtained in the regime of relatively elementary/familiar categories and functors, but where nontrivial and potentially unexpected insights are obtained. For example, once one fixes a model for the notion of an algebraic structure, the monadicity of every forgetful functor from an algebraic structure to a weaker one (or even just Sets) follows easily (e.g. in [11, 1]). One simply verifies that the model possesses the preservation and reflection of U-absolute coequaliser conditions as required by Theorem 6.6, and in doing so "checks the necessary boxes".

We will conclude by sketching the proof of a striking and self-contained result of this type, originally due to Paré ([13]).

Theorem 7.3. *(Paré 1971) The forgetful functor* $U : \mathsf{CptHaus} \to \mathsf{Sets}$ *from the category of compact Hausdorff topological spaces to sets is algebraic(!) in the following sense: U is a monadic functor.*

Proof sketch. We will verify the hypotheses of Theorem 6.6. First, we have a left adjoint $F : \mathsf{Sets} \to \mathsf{CptHaus}$ of U by taking the Stone-Čech compactification of sets equipped with the discrete topology. Morphisms of compact Hausdorff spaces which are bijective on the underlying sets are homeomorphisms, and so U reflects isomorphisms. We show that CptHaus has coequalisers of U-absolute pairs by bootstrapping off the fact that Top does. Indeed, for $(h, h') : X \to X'$ in CptHaus a U-split pair, we get a coequaliser $q : X' \to Y$ (in Top). But $U : \mathsf{CptHaus} \to \mathsf{Sets}$ is the restriction of the forgetful functor $U : \mathsf{Top} \to \mathsf{Sets}$, and the latter preserves all colimits of any kind by the fact that it has a right adjoint (equipping sets with the indiscrete topology).

Therefore it remains to show that Y is compact Hausdorff (technically we need that CptHaus is a full subcategory of Top). Every epimorphism in Top is surjective, and so Y, being the image of a compact space, is compact. A point-set topology argument then shows that we can separate pairs of points in Y with pairs of disjoint open neighbourhoods of those points: we just lift a pair of points to closed sets to their preimage under q, take disjoint open neighbourhoods in the compact Hausdorff space X', send their complements to closed sets back in Y, and finally take complements again. This shows that Y is Hausdorff, and hence completes the proof. □

A final thought; one might be curious as to the action of the monad defined by $U : \mathsf{CptHaus} \to \mathsf{Sets}$, i.e. the effect of the underling composition $T = UF : \mathsf{Set} \to \mathsf{Set}$ on actual sets. The answer (of [10]) is that T sends $X \in \mathsf{Set}$ to the set of *ultrafilters* on the power set of X (yes, really!), partially ordered by inclusion. We have stumbled upon the *ultrafilter monad*, which might just persuade the reader that the notion of an ultrafilter is not so foreign after all!

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