Motivation

A main goal of algebraic topology:

$$\left\{ \text{topological spaces} \right\} \xrightarrow{\;F\;} \left\{ \text{algebraic objects} \right\}$$

$$X \longmapsto F(X)$$

$$\mathbb{R},\ S^1,\ \dots \longmapsto \{e\},\ \mathbb{Z},\ \dots$$

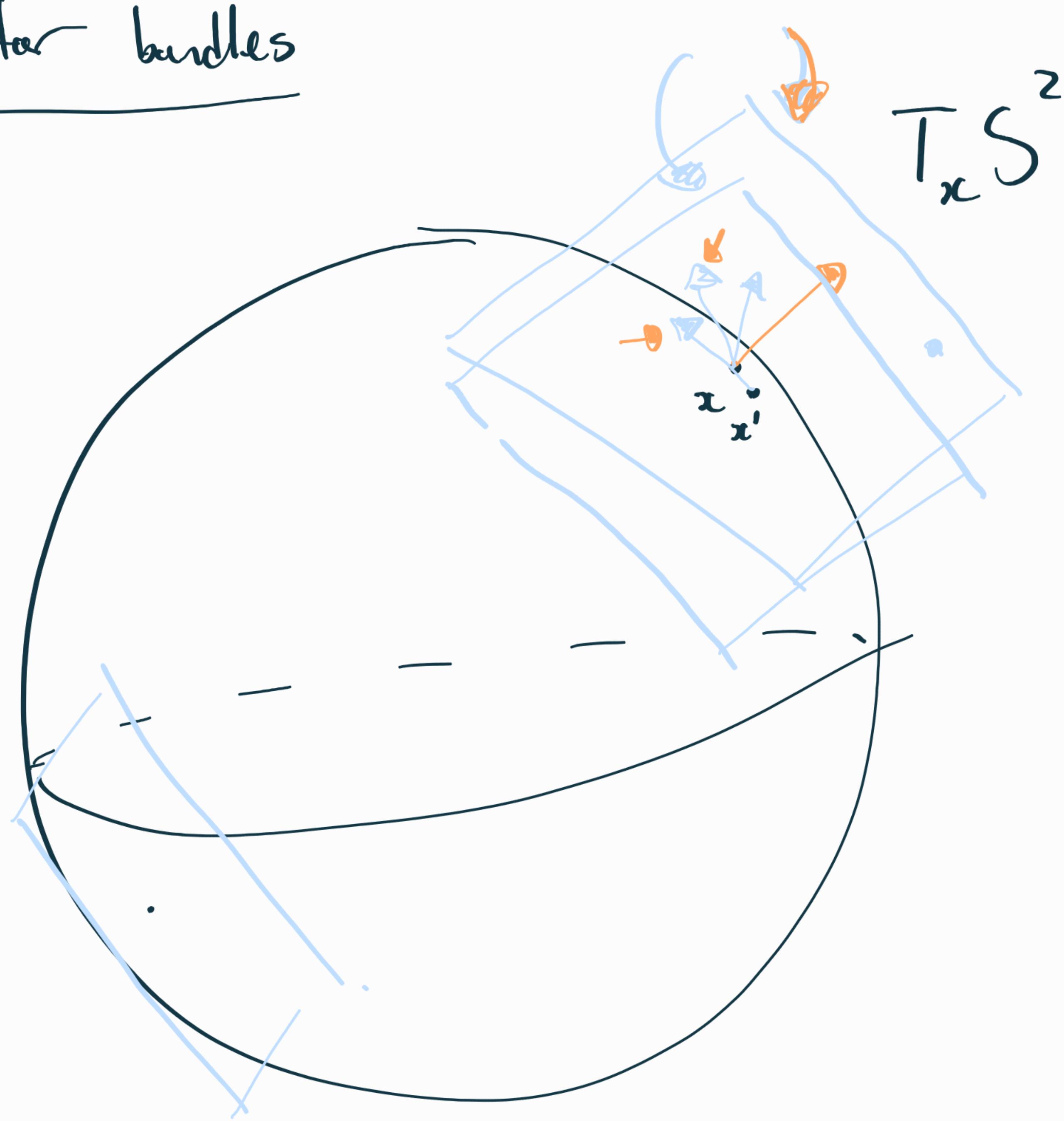
$$\begin{tikzcd}
X \arrow[d, "f"] & \longmapsto & F(X) \arrow[d, "F(f)"] & \text{or} & F(X) \\
Y & & F(Y) & & F(Y) \arrow[u]
\end{tikzcd}$$

This assignment is a (covariant) functor.
(contravariant)

K-theory is such a functor

$$\mathrm{Top} \xrightarrow{K} \mathrm{Ring}.$$

Vector bundles



$$S^2 \subseteq \mathbb{R}^3$$

$$x \longmapsto T_x S^2$$

$$TS^2 := \bigsqcup_{x \in S^2} T_x S^2$$

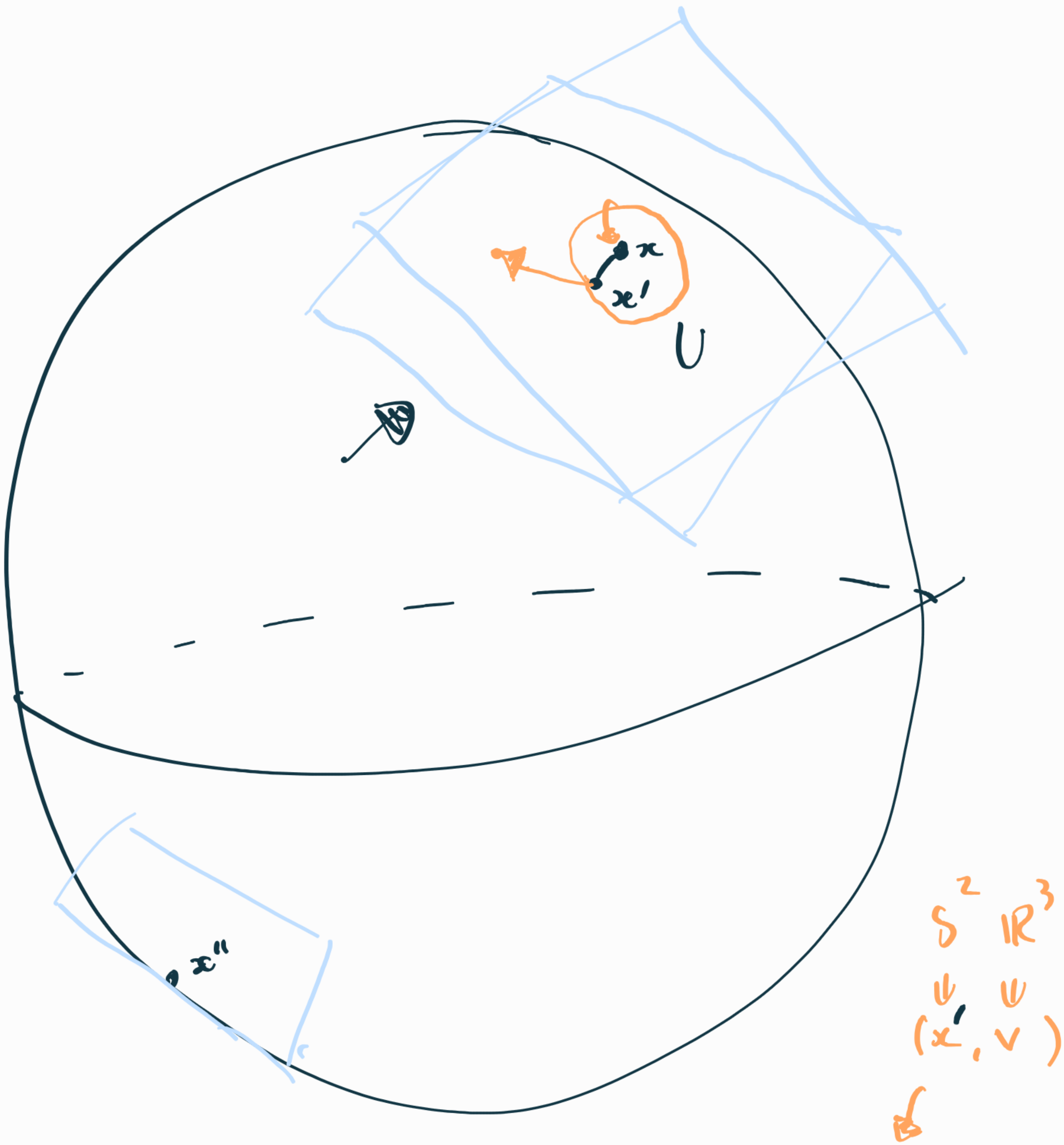
$$\begin{array}{ccccc} TS^2 & \hookrightarrow & S^2 \times \mathbb{R}^3 & \longrightarrow & S^2 \\ \cup & & & & \\ T_x S^2 & & & & \\ \in & & & & \\ v & \longmapsto & (x, v) & & \end{array}$$

<u>Observations</u>: ①

$$\begin{tikzcd} TS^2 \arrow[d, "p"] \\ S^2 \end{tikzcd} \qquad p^{-1}(x) = T_x S^2, \quad x \in S^2.$$

② Each fiber $T_x S^2$ is a vector space.

③ Local triviality



There is $U \subseteq S^2$ so that $p^{-1}(U)$
* is homeomorphic to $U \times \mathbb{R}^2$, which takes
$p^{-1}(x')$ to $\{x'\} \times \mathbb{R}^2 \subseteq U \times \mathbb{R}^2$ via linear iso.

$$TS^2 \supseteq p^{-1}(U) \longrightarrow U \times T_x S^2 \cong U \times \mathbb{R}^2$$

$$(x', v) \longmapsto (x', \mathrm{pr}_{T_x S^2} v)$$

$$\begin{array}{c} TS^2 \\ \cup\!\!| \end{array}$$

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr] \arrow[dr, "p"'] & \arrow[d, phantom, "="] & U \times \mathbb{R}^2 \arrow[dl, "\mathrm{pr}_1"] \\
& U &
\end{tikzcd}$$

$$\begin{tikzcd}
S^2 \times \mathbb{R}^2 \arrow[d, "\mathrm{pr}_1"] \\
S^2
\end{tikzcd}$$

is another example!

Defn. A <u>vector bundle</u> over B (base) is

a (continuous function) map $p: E \longrightarrow B$ such that:

(total space)

① Each fiber $p^{-1}(x)$ is a vector space.

$x \in B$

② (Local triviality)

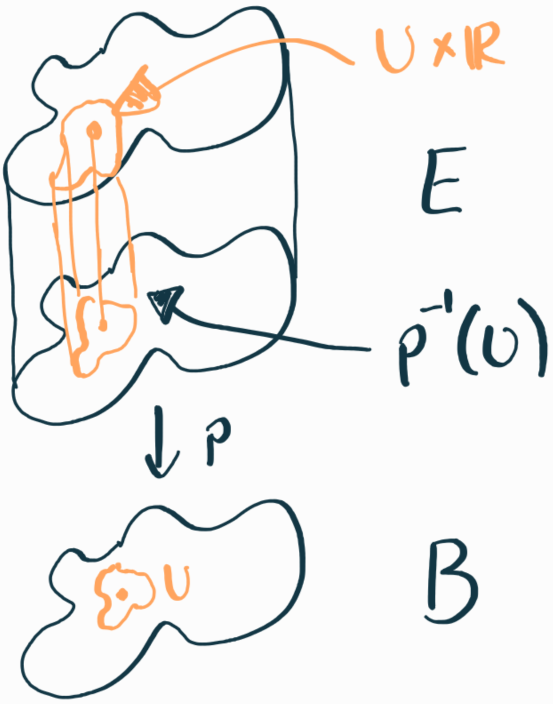
For each $x \in B$ there is an open $U \subseteq B$ containing x and a homeomorphism

(for some n)

$$p^{-1}(U) \longrightarrow U \times \mathbb{R}^n$$

which takes each $p^{-1}(x')$ to $\{x'\} \times \mathbb{R}^n$ via linear isomorphism.

$x' \in U$



Examples

(A) Trivial bundles

$$\begin{array}{c} B \times \mathbb{R}^n \\ \Big\downarrow {\scriptstyle pr_1 =: p} \\ B \end{array}$$

(1) $p^{-1}(x) = \{x\} \times \mathbb{R}^n$ ✓

(2) $B \times \mathbb{R}^n \xrightarrow{id} B \times \mathbb{R}^n$ ✓

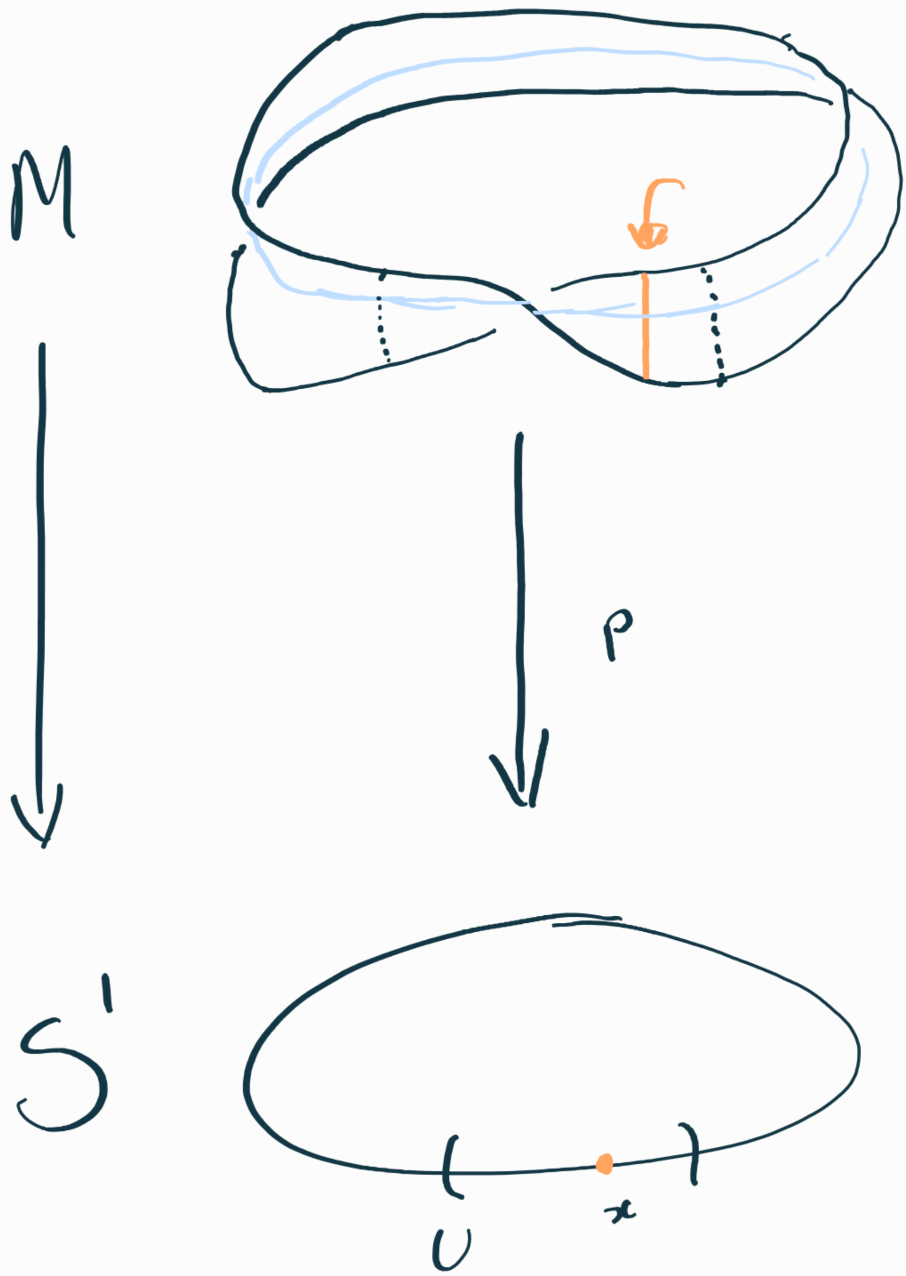
(B) Vector spaces

$$\begin{array}{c} V \\ \Big\downarrow {\scriptstyle p} \\ \{\bullet\} \end{array}$$

(1) $p^{-1}(\bullet) = V$ ✓

(2) $p^{-1}(\bullet) = V \xrightarrow{\downarrow} \{\bullet\} \times \mathbb{R}^n$ ✓

(C) Möbius bundle

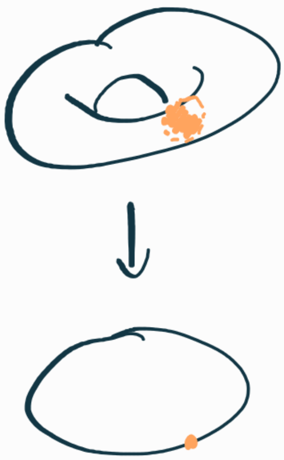


① $p^{-1}(x) \cong \mathbb{R}$

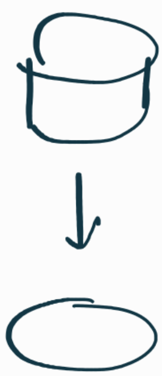
② Local triv?

$U \times \mathbb{R}$

(D) $S^1 \times \mathbb{R}^2$



$S^1 \times \mathbb{R}$



Map of vector bundles

Defn. A map $\begin{array}{c} E \\ \downarrow p \\ B \end{array} \longrightarrow \begin{array}{c} F \\ \downarrow q \\ B \end{array}$ is a map $E \xrightarrow{f} F$ such that:

①
$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & F \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

f takes the fiber of E above $x \in B$ $(p^{-1}(x))$ to $q^{-1}(x)$.

② Each restriction

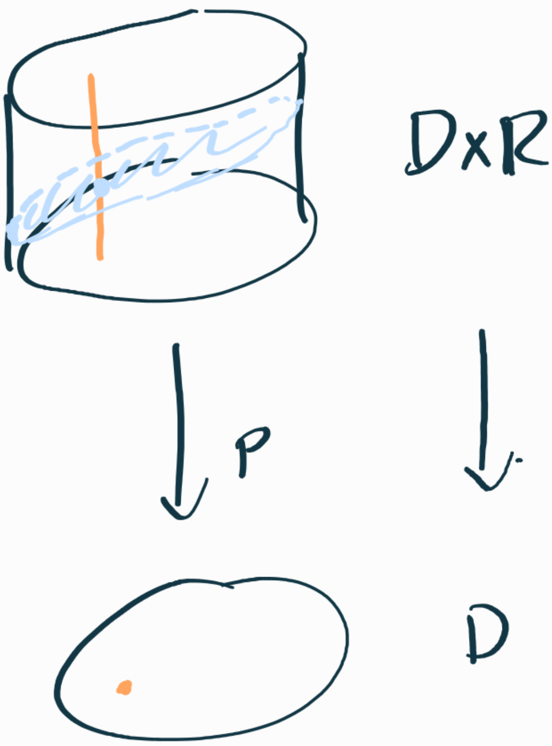
$$\begin{tikzcd} f : p^{-1}(x) \arrow[r] \arrow[d, phantom, sloped, "\subseteq"] & q^{-1}(x) \arrow[d, phantom, sloped, "\subseteq"] \\ E & F \end{tikzcd}$$

is a linear map.

Notation: $E_x := p^{-1}(x)$ $\quad E_U := p^{-1}(U)$.

Q. Is there an isomorphism $M \xrightarrow{f} S^1 \times \mathbb{R}$?

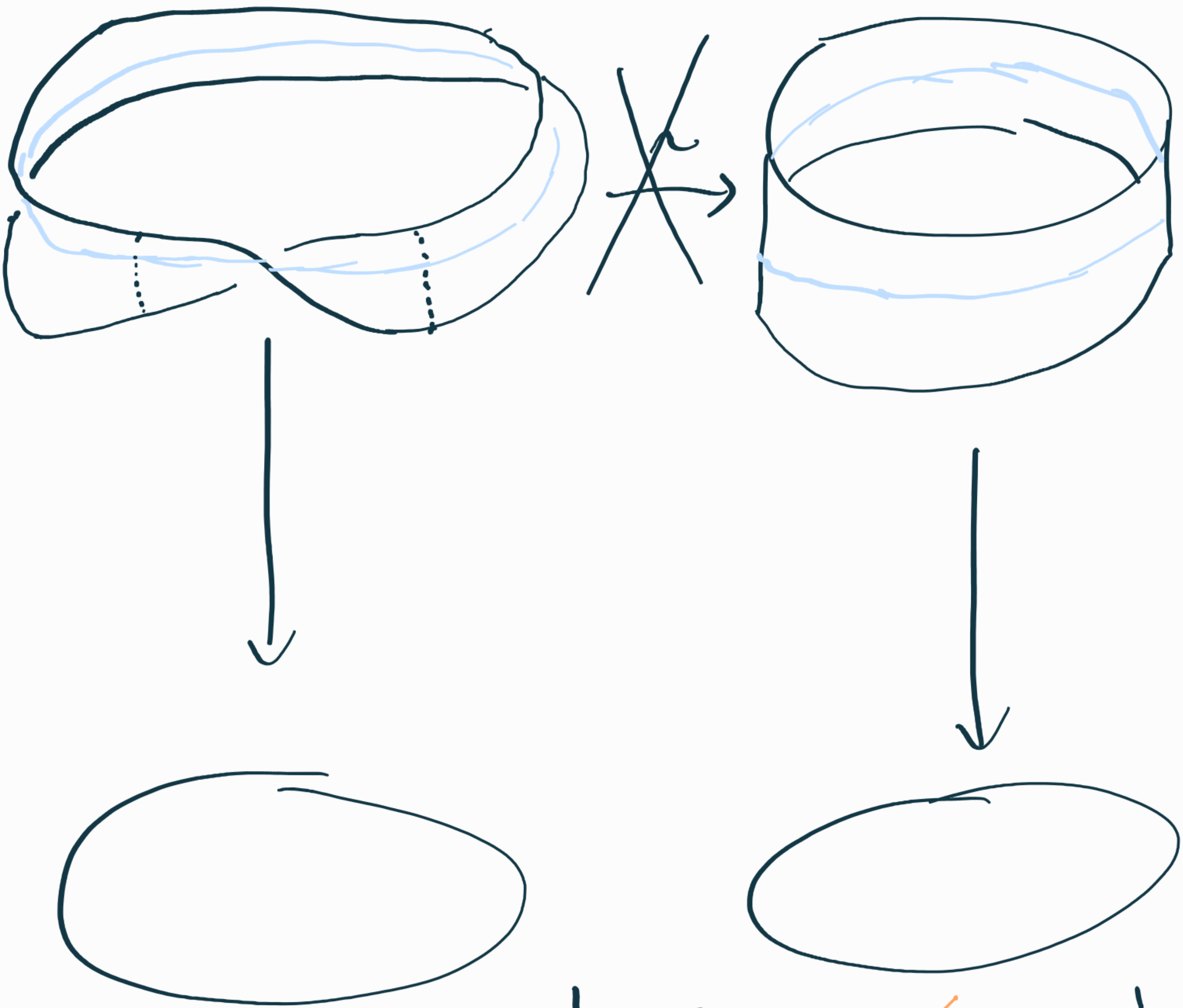
Defn. A section σ of a v.b. $\begin{matrix} E \\ \downarrow p \\ B \end{matrix}$ is a map $B \xrightarrow{\sigma} E$ such that $p \circ \sigma = id_B$.



Observation. Every vector bundle $\begin{matrix} E \\ \downarrow p \\ B \end{matrix}$ has the zero section defined by

$$\begin{matrix} x & \longmapsto & 0 \\ \cap & & \cap \\ B & & E_x (= p^{-1}(x)) \end{matrix}.$$

M $\qquad\qquad\qquad$ $S^1 \times \mathbb{R}$



$S^1 \times \mathbb{R}^2$.

Ex $TS^1 \cong S^1 \times \mathbb{R}$

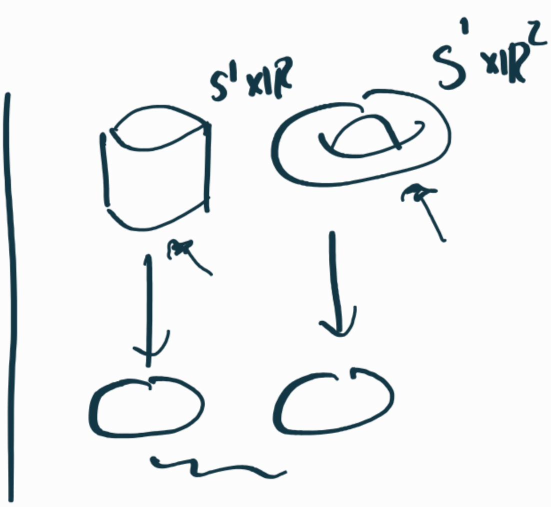
Also $TS^2 \not\cong S^2 \times \mathbb{R}^2$

$TS^3 \cong S^3 \times \mathbb{R}^3$

$TS^4 \not\cong S^4 \times \mathbb{R}^4$

$\vdots$

$TS^7 \cong S^7 \times \mathbb{R}^7$



Defn. If every fiber of $\begin{matrix} E \\ \downarrow p \\ B \end{matrix}$ has dimension n then we say E is rank n.

Prop. A vector bundle of rank n is trivial iff there exist n everywhere linearly-independent sections $\sigma_1, \dots, \sigma_n$.

$$\begin{tikzcd} E \arrow[d, shift left, "p"] & \sigma(x) \in E_x \arrow[l] \\ B \arrow[u, shift left, "\sigma"] & x \arrow[l] \end{tikzcd}$$

$\sigma_1(x), \dots, \sigma_n(x).$

Pf. ($\Rightarrow$) Suppose $\begin{matrix} E \\ \downarrow p \\ B \end{matrix}$ is trivial and so there is an iso $\phi : E \longrightarrow B \times \mathbb{R}^n$.

Observe that if σ is a section of $\begin{matrix} B \times \mathbb{R}^n \\ \downarrow \\ B \end{matrix}$ then $\phi \circ \sigma$ is a section of E.

We have n, everywhere linearly-indep. sections σ_i of $B \times \mathbb{R}^n$ given by

$$\begin{array}{ccc} x & \longmapsto & (x, e_i) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ B & & B \times \mathbb{R}^n \end{array}$$

(where $\{e_i\}$ is the standard basis of $\mathbb{R}^n$.)

($\Longleftarrow$) Suppose that there are n everywhere linearly-indep. sections $\sigma_1, \ldots, \sigma_n$ of E.

We are supposed to produce a map

$$\underset{\sim}{\phi} : E \longrightarrow B \times \mathbb{R}^n.$$

Given $v \in E_x$, $\{\sigma_1(x), \ldots, \sigma_n(x)\}$ is a basis for E_x, and so there are n uniquely determined $c_1, \ldots, c_n$ such that

$$v = c_1 \sigma_1(x) + \cdots + c_n \sigma_n(x).$$

Let's define $\phi(v) := (x, (c_1, \ldots, c_n))$.

On the other hand define

$$\psi : B \times \mathbb{R}^n \longrightarrow E$$

by $\underset{\sim}{\psi}(x, (c_1, \ldots, c_n)) := c_1 \sigma_1(x) + \cdots + c_n \sigma_n(x)$.

The claim now follows from a general Fact. $\boxtimes$

Prop. A map $\begin{array}{c}E\\ \downarrow\\ B\end{array} \xrightarrow{f} \begin{array}{c}F\\ \downarrow\\ B\end{array}$ which is an isomorphism on each fiber is an isomorphism.

Pf. To be done together next time. $\boxtimes$

Use case. $\begin{array}{c}E\\ \downarrow p\\ B\end{array}$, $\begin{array}{c}F\\ \downarrow q\\ B\end{array}$, we'd like to define $\begin{array}{c}E \oplus F\\ \downarrow\\ B\end{array}$.

$\to$ W

You might wonder: what about vector bundles over other bases?

e.g. $\begin{array}{c}E\\ \downarrow\\ B\end{array}$ $\begin{array}{c}F\\ \downarrow\\ C\end{array}$

A: It turns out that all of this complexity is subsumed by having the same base.

Observations. Starting with a v.b. $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$,

there is for each $x \in B$ an open $U_x \subseteq B$, containing x, and there is an iso

$$\begin{tikzcd} p^{-1}(U_x) = E_{U_x} \arrow[r, "\phi_{U_x}", "\sim"'] \arrow[d] & U_x \times \mathbb{R}^n \arrow[dl] \\ U_x & \end{tikzcd}$$

Can we recover E? Yes!

$$\tilde{E} := \coprod_{x \in B} U_x \times \mathbb{R}^n \Big/ \sim$$

$$\underset{U_x \times \mathbb{R}^n}{(y,v)} \sim \underset{U_{x'} \times \mathbb{R}^n}{(y',v')} \text{ exactly}$$

when $\phi_{U_x}^{-1}(y,v) = \phi_{U_{x'}}^{-1}(y',v')$

Ex. Show this is an equiv. relation.

We get $\hat{p} : \tilde{E} \longrightarrow B$ by

$$U_x \times \mathbb{R}^n \ni (y,v) \longmapsto y.$$

↳ We need one more thing — cocycle condition.