<u>Ex</u>. Let's calculate the K-groups of $\mathbb{R}P^2$ & $\mathbb{R}P^4/\mathbb{R}P^2$.

(For $\mathbb{R}P^2$ we consider $\mathbb{R}P^1 \subseteq \mathbb{R}P^2$.)

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, phantom, "\shortparallel"] & & 0 \\
\tilde{K}^0(\mathbb{R}P^2/\mathbb{R}P^1) \arrow[r] & \tilde{K}^0(\mathbb{R}P^2) \arrow[r] & \cancel{\tilde{K}^0(\mathbb{R}P^1)} \arrow[d] \\
\tilde{K}^1(\mathbb{R}P^1) \arrow[u, "2"] \arrow[d, phantom, "\shortparallel"] & \tilde{K}^1(\mathbb{R}P^2) \arrow[l] & \tilde{K}^1(\mathbb{R}P^2/\mathbb{R}P^1) \arrow[l] \arrow[d, phantom, "\shortparallel"] \\
\mathbb{Z} & & 0
\end{tikzcd}$$

$\mathbb{Z}$ = $\mathbb{Z}\langle H-1\rangle$

$\mathbb{R}P^2/\mathbb{R}P^1 = S^2$

$$0 \to \tilde{K}^1(\mathbb{R}P^2) \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \twoheadrightarrow \tilde{K}^0(\mathbb{R}P^2) \to 0$$

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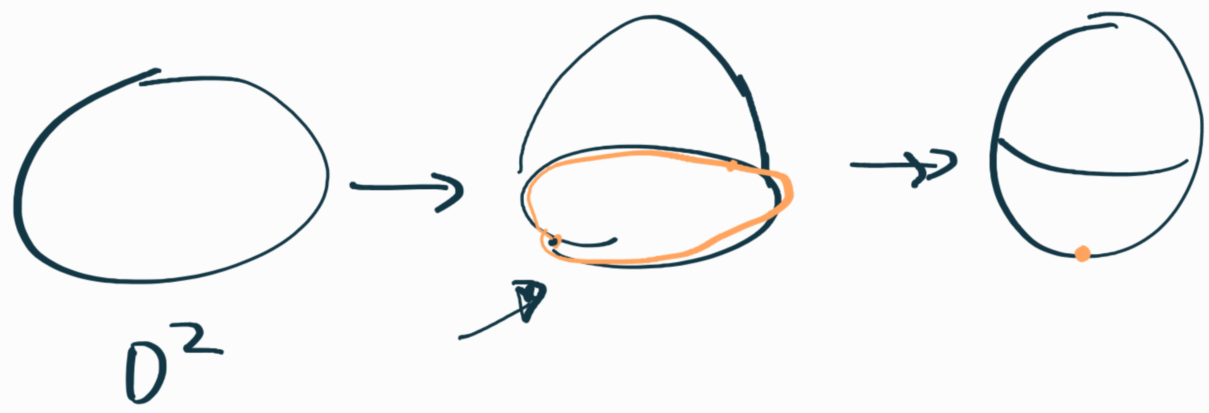
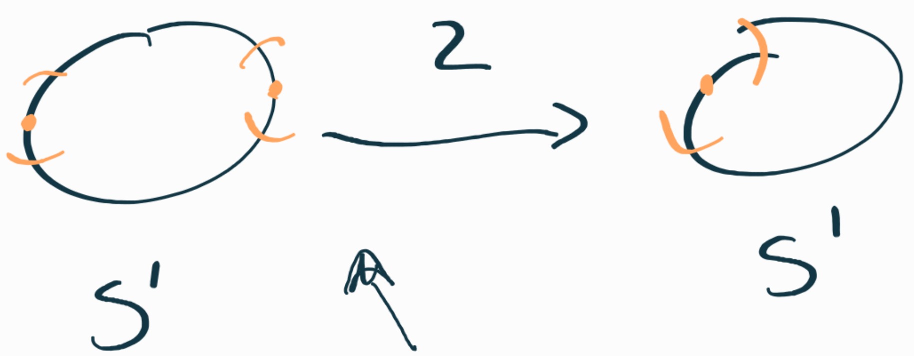
$\tilde{K}^1(\mathbb{R}P^2) = 0 \qquad \tilde{K}^0(\mathbb{R}P^2) = \mathbb{Z}_2$.





Lemma. Suppose $f : S^n \to S^n$ is a cts. map. which for some $x_0 \in S^n$ the preimage $f^{-1}\{x_0\}$ is finite. Moreover, suppose each $x_i \in f^{-1}\{x_0\}$ is contained in an open subset $U_i \subseteq S^n$ with the U_is all disjoint.

If f is a homeomorphism on each

V_i, the degree of f is the sum $\sum_i \sigma_i$, $\sigma_i \in \{-1, 1\}$, $\sigma_i = \begin{cases} 1 & f \text{ is orientation preserving on } V_i \\ -1 & \text{otherwise} \end{cases}$

Ex. Try this again and compute the K-groups of $\mathbb{R}P^4 / \mathbb{R}P^2$.

(Consider $(\mathbb{R}P^4/\mathbb{R}P^2, \mathbb{R}P^3/\mathbb{R}P^2)$.)

$\boxed{\textbf{Ex. } \text{Also try } \mathbb{R}P^4.}$

Hint. Consider the map $\mathbb{R}P^5 \to \mathbb{C}P^2$ induced by $\mathbb{R}^6 \to \mathbb{C}^3$.

You may use that $\tilde{K}^0(\mathbb{C}P^2) \cong \mathbb{Z}[x]/(x^3)$.

Defn. $\tilde{K}^*(X) := \tilde{K}^0(X) \oplus \tilde{K}^1(X)$.

Observe that we have a multiplication

$$\tilde{K}^*(X) \otimes \tilde{K}^*(Y) \longrightarrow \tilde{K}^*(X \wedge Y).$$

external product.

Moreover, when $X=Y$ we also have $\tilde{K}^*(X \wedge X) \longrightarrow \tilde{K}^*(X)$ induced by $X \to X \wedge X$.

Therefore we get $\tilde{K}^*(X) \otimes \tilde{K}^*(X) \longrightarrow \tilde{K}^*(X)$.

Prop. This gives $\tilde{K}^*(X)$ the structure of a ring extending the ring structure on $\tilde{K}^0(X)$.

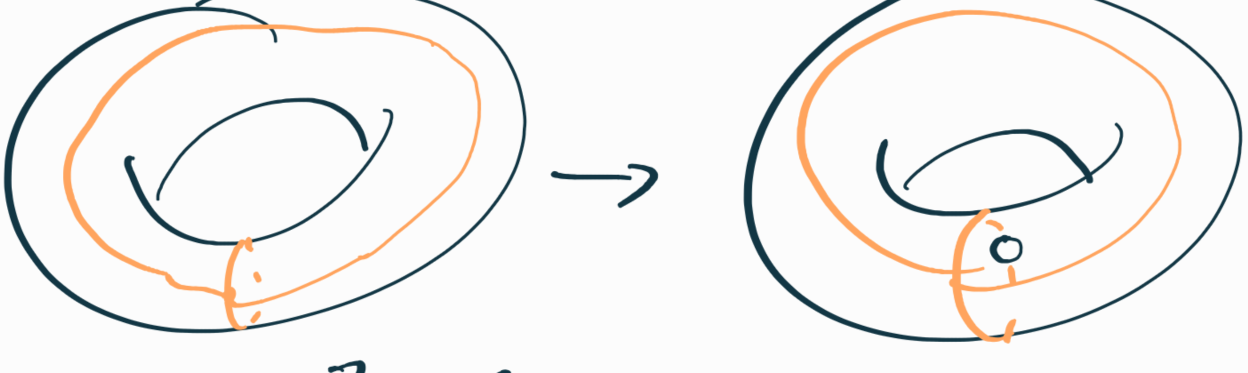
Prop. The ring $\tilde{K}^*(X)$ is graded-commutative, i.e. $ab = (-1)^{|a||b|} ba$ $a \in \tilde{K}^{|a|}(X)$, $b \in \tilde{K}^{|b|}(X)$.

Pf (Sketch). The multiplication on the $\tilde{K}^1(X)$ involves a suspension (in the defn), and commuting classes in $\tilde{K}^*(Y)$ amounts to the "commutativity" map

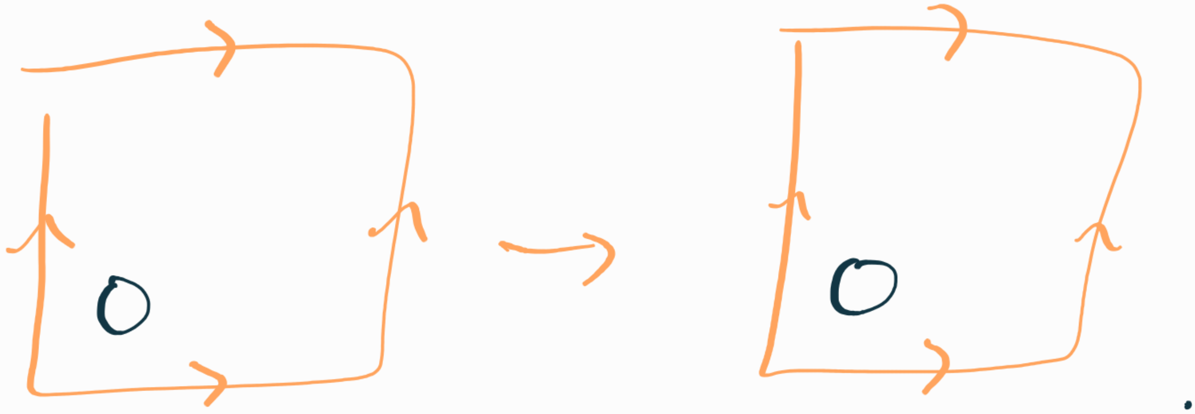
$$S^1 \wedge S^1 \longrightarrow S^1 \wedge S^1. \quad (*)$$

$$\begin{array}{ccc} S^1 \wedge S^1 & \longrightarrow & S^1 \wedge S^1 \\ \cong & & \cong \\ S^2 & & S^2 \end{array}$$

Ex. This map (*) is (up to homotopy) the antipodal map on S^2.



$$T^2 \xrightarrow{\text{swap}} T^2$$



Prop. The 6-term exact sequence carries over to the non-reduced setting if we define:

$$K^*(X) = \tilde{K}^*(X \sqcup \bullet) = \tilde{K}^*(X_+).$$

$$K^*(X,A) = \tilde{K}^*(X,A).$$

(Observation:
$$\underline{K^1(X)} = \tilde{K}^1(X_+) = \tilde{K}(SX_+) = \tilde{K}(S^1 \wedge (X \sqcup \bullet)) = \tilde{K}(S^1 \wedge X \sqcup \cancel{S^1}) = \underline{\tilde{K}^1(X)}$$
).

Prop. This $\longrightarrow$.

$$\left[\begin{tikzcd}[column sep=small] K^*(X,A) \arrow[rr] & & K^*(X) \arrow[dl] \\ & K^*(A) \arrow[ul] & \end{tikzcd}\right.$$

is an exact sequence of $K^*(X)$-modules (and likewise for the reduced case).

Observe that $X/A \longrightarrow X \wedge X/A$

induces a multiplication of $K^*(X)$ on $K^*(X,A)$.

The Fundamental Product Theorem.

$$K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \longrightarrow K(X \times S^2)$$

is an isomorphism.

Observe that given a vector bundle isomorphism

$$\begin{array}{ccc} E \times S^1 & \xrightarrow{f} & E \times S^1 \\ p \downarrow \quad \downarrow id & & \downarrow \\ X \times S^1 & & X \times S^1 \end{array}$$

we can glue two copies of bundle along f and produce a new v.b. over $X \times S^2$.

$$\begin{array}{c} E \times D^2 \\ \downarrow \\ X \times D^2. \end{array}$$

We'll denote the resulting bundle by

$[E, f]$.

Basic examples.

① $[E, \mathrm{id}] = \begin{array}{c} E \times S^2 \\ p\downarrow \ \ \downarrow \mathrm{id} \\ X \times S^2 \end{array} = E \ast \begin{pmatrix} \varepsilon^1 \\ \downarrow \\ S^2 \end{pmatrix}$.

② $\left[\begin{array}{c} \varepsilon^1 \\ \downarrow \\ \bullet \end{array}, z \right] = \begin{array}{c} H \\ \downarrow \\ \mathbb{C}P^1 \end{array}$, $\left[\begin{array}{c} \varepsilon^1 \\ \downarrow \\ \bullet \end{array}, z^n \right] = H^{\otimes n}$.

③ $[E, z^n] = E \ast (H^{\otimes n})$

④ If f is any generalized clutching function for E, then $z^n f$ always makes sense as another gen. clutching function. We have the identity

$$[E, z^n f] \cong [E, f] \otimes \hat{H}^{\otimes n}$$

where $\hat{H} = (pr_2^*) H$.

$$\begin{tikzcd} X \times S^2 \arrow[d, "pr_2"] \\ S^2 \end{tikzcd}$$

(Step 0)

Prop. Every vector bundle $\begin{matrix} E \\ \downarrow \\ X \times S^2 \end{matrix}$ is isomorphic to $[E', f]$ for some E' and gen. clutching function f.

Pf. Decompose S^2 as $D^2_+ \cup D^2_-$ with $D^2_+ \cap D^2_- = S^1$ and form restrictions

$$\begin{matrix} E|_{D^2_\pm} & & E_{1 \in S^1} \\ \downarrow & & \downarrow \\ X \times D^2_\pm & & X \times \{1\}. \end{matrix}$$

We can now pick "trivializations"

fiberwise isomorphism

$$h_\pm : E|_{D^2_\pm} \longrightarrow E_1$$

coming from the fact that $D^2_\pm$ deformation retracts onto $1 \in S^1$.

$$E|_{D^2_+} \xrightarrow{h_+} E_1 \xrightarrow{h_-^{-1}} E|_{D^2_-}$$ makes

sense over $X \times S^1$, and is a vector bundle isomorphism.

We can thus produce $\left[E_1, \; h_-^{-1} \circ h_+ |_{X \times S^1}\right]$.

$\boxed{\text{}}$

Step (D).

Prop. Every $[E, F]$ (arbitrary) is isomorphic to $[E, \ell]$ where ℓ is Laurent clutching function.

$$\ell(x, z) = \sum_{i \le |n|} a_i(x) z^i$$

E S^1 — an endo. of E.

Lemma. Every map $X \times S^1 \to \mathbb{C}$ (cpt. Hausdorff.) is uniformly approximable by Laurent polynomial functions. (ie. of the form $\sum_{i \le |n|} a_i(x) z^i$ — functions on X.)

Pf. Appears in Hatcher. Just use Fourier series. ∎

Pf. of Prop. Starting with $[E, f]$, pick an inner product on $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$. Observe that the space of vector bundle maps $E \to E$ $(\mathrm{End}(E))$ is a normed space when equipped with the norm

$$\|g\| = \sup_{x \in X} \|g(x)\|,$$

and likewise for $E \times S^1$ as well.

1 Also $\mathrm{Aut}(E \times S^1) \subseteq \mathrm{End}(E \times S^1)$ is an open subset, since

$\inf_{x \in X} |\det g(x)|$ is a well-defined continuous map $\mathrm{End}(E \times S^1) \to \mathbb{R}$ whose preimage of $(0, \infty)$ is just $\mathrm{Aut}(E \times S^1)$.

(2) We have that $\|\cdot\|$ on $\mathrm{End}(E \times S^1)$ obeys the triangle inequality, so that if $f \in \mathrm{Aut}(E \times S^1)$ then for all $g \in \mathrm{Aut}(E \times S^1)$ the convex combination $tf + (1-t)g$ is also in $\mathrm{Aut}(E \times S^1)$.

$\hookrightarrow$ It follows that it is enough to show that the Laurent poly. clutching functions are dense in $\mathrm{End}(E \times S^1)$,

Since by (2), f would then be homotopic to some Laurent clutching function.

Pick open subsets U_i covering X such that X is trivial $\{U_i\}$, and so also choose isos,

$$h_i : E|_{U_i} \xrightarrow{\sim} U_i \times \mathbb{C}^n,$$

Recall that we may assume h_i is inner product preserving.

Also choose $\{\varphi_i\}$ a part. of unity subordinate to the cover $\{U_i\}$ and let $X_i = \overline{\varphi_i^{-1}(0,1]} \subseteq U_i$ be the respective supports. Conjugation by the h_is gives a matrix formula for f on each X_i, and each matrix-entry projection in turn gives a function $f_{ijk} : X_i \times S^1 \to \mathbb{C}$ for

f over U_i.

Apply the lemma and obtain a sequence of Laurent polynomial functions approximating f_{ijk} uniformly arbitrarily well.

f_{ijk}^{m}

There are only finitely many of these, so the Laurent polynomial clutching function they together define (for good enough approx.)

$$\ell_m(x,z) = \sum_{U_i} \sum_{|n| \le N_m} \varphi_i(x) \begin{pmatrix} \text{Laurent} \\ \text{polynomial} \end{pmatrix} \text{endomorphism}$$

can be made to approximate f arbitrarily well. $\square$

Step (1.5) Note then that every

$$[E,f] \overset{\sim}{=} [E, z^{-m}\hat{f}] = \underbrace{[E,\hat{f}]}\otimes \underbrace{\hat{H}^{\otimes m}}.$$

for $\hat{f}$ polynomial.

Step (2) Reduction further to the case of linear clutching functions

(i.e. $\hat{f} = a(x) z + b(x)$.)

$x \in X$

Prop. We have $[E, \tilde{f}] \oplus [nE, \mathbb{1}]$ (polynomial) $n = \deg f.$

$$\overset{\sim}{=} [(n+1)E, \hat{f}]$$ linear

$\hat{f} = a_0(x) + a_1(x)z + \dots + a_n(x)z^n$

Pf. The LHS is glued via

$$\begin{bmatrix} 1 & -z & & & \\ & 1 & -z & & \\ & & 1 \ddots & -z & \\ & & & 1 & -z \\ a_n & a_{n-1} & a_{n-2} \cdots & a_1 & a_0 \end{bmatrix}$$

while the RHS is glued via.

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \tilde{f} \end{bmatrix}$$

(in the sense that they are endomorphisms of $(n+1)E$ where we interpret the (i,j)-entry as a bundle map from the ith factor to the jth.

... To be continued.