Recap. ⓪ $\begin{matrix} E \\ \downarrow \\ X \times S^2 \end{matrix} \cong [\begin{matrix} E' \\ \downarrow \\ X \end{matrix}, f]$

① $[E, f] \cong [E, \ell]$ Laurent polynomial clutching func.

② $[E, \ell] \cong [E, q] \otimes \hat{H}^{-m}$ polynomial

$(H-1)^2 = 0$.

③ $[E, q] \oplus [nE, id] \cong [(n+1)E, a(x)z + b(x)]$ ($n \geq \deg q$.) Linear.

↳ Resume.

④ $[E, a(x)z + b(x)] \cong [E, z + b'(x)]$

⑤ For any $[E, z + b(x)]$ $\exists! \tilde{E} \cong E_+ \oplus E_-$

$[E, z + b(x)] \cong [E_+, id] \oplus [E_-, z]$.

Pf of surj of FPT. For an arbitrary $E' = [E, f] \in K(X \times S^2)$

we have. (in $K(X \times S^2)$)

$$
\begin{aligned}
\underset{\sim}{[E, f]} &= [E, \underset{\sim}{q}] \otimes \hat{H}^{-m} \\
&= [(n+1)E, \underbrace{a(x)z + b(x)}_{\overset{\shortparallel}{z + b'(x)}}] \otimes \hat{H}^{-m} \\
&\qquad - [nE, \mathrm{id}] \otimes \hat{H}^{-m} \\
&= [((n+1)E)_{+}, \mathrm{id}] \otimes \hat{H}^{-m} \\
&\qquad + [((n+1)E)_{-}, \overset{\mathrm{id}}{\cancel{\underset{\sim}{z}}}] \otimes \hat{H}^{-m+1} \\
&\qquad - [nE, \mathrm{id}] \otimes \hat{H}^{-m} \\
&= \mu\Big(\underset{\underset{K(X)}{\Uparrow}}{((n+1)E)_{+}} \otimes \underset{\underset{K(S^2)}{\Uparrow}}{H^{-m}} + ((n+1)E)_{-} \otimes H^{-m+1} \\
&\qquad\qquad - nE \otimes H^{-m}\Big).
\end{aligned}
$$

i.e. μ is surjective.

∎

Injectivity will instead follow from unwrapping these constructions, so we'll have to see them first...

Pf. of (3).

We could define $[nE, \underset{\sim}{id}] \oplus [E, \underset{\sim}{q}]$ by using the trivial clutching function

arb. linear clutching func.

$$\underset{\sim}{g_1} = \begin{array}{c} \begin{array}{ccccc} E & \cdots & & E & E \end{array} \\ \left[\begin{array}{cccc|c} 1 & & & & 0 \\ & \ddots & & & \\ & & \ddots & & \\ & & & 1 & \\ \hline & & & & \underset{\sim}{q} \end{array}\right] \end{array}$$

the (i,j)-entry of this matrix is a bundle map $E_i \to E_j$

for $(n+1)E$.

Claim. By a sequence of row ops, and then col. ops, we can produce from g_1 the clutching func.

$$g_z = \begin{bmatrix} 1 & -z & & & \\ & 1 & -z & & \\ & & \ddots & \ddots & \\ & & & 1 & -z \\ a_n & \cdots & \cdots & a_1 & a_0 \end{bmatrix}$$

$$\left(q(x,z) = a_0(x) + a_1(x) z + \cdots + a_n(x) z^n \right)$$

Now observe that g_z is of the form $a(x)\, z + b(x)$; and row & column operations are implementable by homotopy (through clutching functions).

$\blacksquare$

Note that for fixed $n \geq \deg q$, we wrote down a recipe to build a linear clutching function L^n_q from q.

Pf of ④. Cheap trick: ($q(x,z) = a(x)z + b(x)$).

Consider the family of maps

$$q_t(x,z) = a(x)(z+t) + b(x)(1+zt)$$

↳ when $t=0$ of course $q_0 = q$.

Moreover, for any $0 \le t_0 < 1$, then

$$[E,q] \cong [E,q_t] \quad \forall t \le t_0,$$

since

always nonzero when $z \in S^1$ & $t<1$

$$q_t(x,z) = (1+zt)\cdot q\left(\frac{1}{z}\frac{z+t}{\bar{z}+t}\right)$$

sends S^1 into itself.

Observe that $a(x)+b(x)$ is an iso.
(Follows just since we can restrict q to $X \times \{1\}$.)

In turn, for all t_0 close enough to $1 \in I$, $a(x) + t\,b(x)$ must still be invertible. → The map

$$t \mapsto \inf_{x \in X} \left| \det\left(a(x) + t\, b(x)\right) \right|$$

is continuous and strictly positive at $t=1$.

Pick any such t_0. Then

$$q_{t_0}(x,z) = \cancel{a(x)}z + a(x)\,t_0 + b(x) + \cancel{b(x)}z\,t_0$$

$$= \left(a(x) + t_0 b(x)\right) z + a(x) t_0 + b(x).$$

$$= \left(a(x) + t_0 b(x)\right)\Big(\underbrace{z + \left(a(x) + t_0 b(x)\right)^{-1} \left(a(x) t_0 + b(x)\right)}_{\tilde{q}_{t_0}(x,z)}\Big).$$

$$[E, q] \cong [E, q_{t_0}]$$

$$\cong \Big[E, \underbrace{(a(x)+t_0 b(x))}\, \hat{q}_{t_0}\Big]$$

$$\cong \big[E, \hat{q}_{t_0}\big].$$

Q. If $\alpha \in GL(\mathbb{C}^n)$, then $S^1 \ni z \longmapsto \alpha$ defines an "ordinary" clutching func. for a bundle over S^2.

What is this bundle? Always the trivial bundle

$$\begin{array}{c} \mathbb{C}^n \times S^2 \\ \downarrow \\ S^2 \end{array}$$

For our generalized clutching functions we always have

$$[E, \underset{\substack{\uparrow \\ \mathrm{Aut}(E)}}{\alpha} f] \cong [E, f] \qquad \blacksquare$$

For right now, note that if $z + b(x)$ is a clutching function, then b has no eigenvalues in $S^1 \subseteq \mathbb{C}$.

<u>Pf of</u> Ⓢ.

<u>Lemma</u>. Suppose $\underline{b : E \to E}$ is an endomorphism with no e.v.s in S^1. Then E uniquely decomposes as $E_+ \oplus E_-$ such that:

$\boxed{A}$ b respects the decomp.

$\boxed{B}$ evs.$(b|E_+)$ all lie outside the unit circle, & evs.$(b|E_-)$ all lie inside the unit circle.

Pf of lemma. Let's do this for v. spaces first; Fix $T: V \longrightarrow V$, let $q(t)$ be the char. poly. of T.

Define $q_+(t)$ = "product of all linear factors of $q(t)$ which have roots outside $S^1 \subset \mathbb{C}$",

$q_-(t)$ = " ______________
____
______ inside $S^1 \subset \mathbb{C}$".

Then $q(t) = q_+(t)\, q_-(t)$,

Define $V_\pm := \ker q_\pm(T)$. First, q_+ and q_- are coprime, so $\exists$ polys. r & s s.t. $rq_+ + sq_- = 1$. In particular

① $r(T)q_+(T) + s(T)q_-(T) = \mathrm{id}_V$

$0 = q(T) = q_+(T)q_-(T) = q_-(T)q_+(T)$ $\Rightarrow \operatorname{im} q_-(T) \subseteq V_+ = \ker q_+(T)$

$\Rightarrow V_+ = \operatorname{im} q_-(T).$

② $q_+(T)r(T) + q_-(T)s(T) = \mathrm{id}_V.$

$\Rightarrow \operatorname{im} q_+ + \operatorname{im} q_- = V.$

$\Rightarrow V_+ + V_- = V \quad (*)$

Q: Why do we know

$V_+ \cap V_- = \phi \quad (**)$?

↳ Immediately follows from ②.

$q_+(T)Tv = Tq_+(T)v = 0 \Rightarrow T(V_+) \subseteq V_+.$

$v \in V_+ = \ker q_+(T)$

This shows $\boxed{A}$. B_y is easier.

To see uniqueness let $V = V_+ \oplus V_-$ be an arb. decomp. of V satisfying $\boxed{A}$ & $\boxed{B}$.

we can restrict $b_\pm : V_\pm \to V_\pm$, and obtain char. polys q'_+, q'_- respectively. $\to q = q'_+ \cdot q'_-$.

But by $\boxed{B}$ we must have that the factors of $q'_\pm$ are factors of $q_\pm$.

We have to have that $q'_\pm(T)$ annihilates $V_\pm$, which implies that $V_\pm \subseteq \ker q_\pm(T)$. But by $\boxed{A}$ $V = V_+ \oplus V_-$, so the only possibility is that we have equality.

Claim. The polys. q_+ & q_- vary continuously as a function of T.

Pf of claim:

Let $a \in \mathbb{C} / S^1$ be a root of q with multiplicity m.
Let C be a small circle in $\mathbb{C}$ around
"$\{a + re^{i\theta} : \theta \in [0, 2\pi)\}$"
a. If q' is

$\hookrightarrow$ ⊕

As we vary our point in the base a small amount and q becomes q' then

Subclaim. The number of roots of q inside a circle C (disjoint from the zero locus of q) is equal to the winding number of the map

$\{a + re^{i\theta}\}$

$$\underset{\substack{\cap \\ C}}{\gamma(z)} = \frac{q(z)}{|q(z)|}$$

$$C \longrightarrow S^1.$$

Pf. Let a be any root of q of multiplicity m. Then

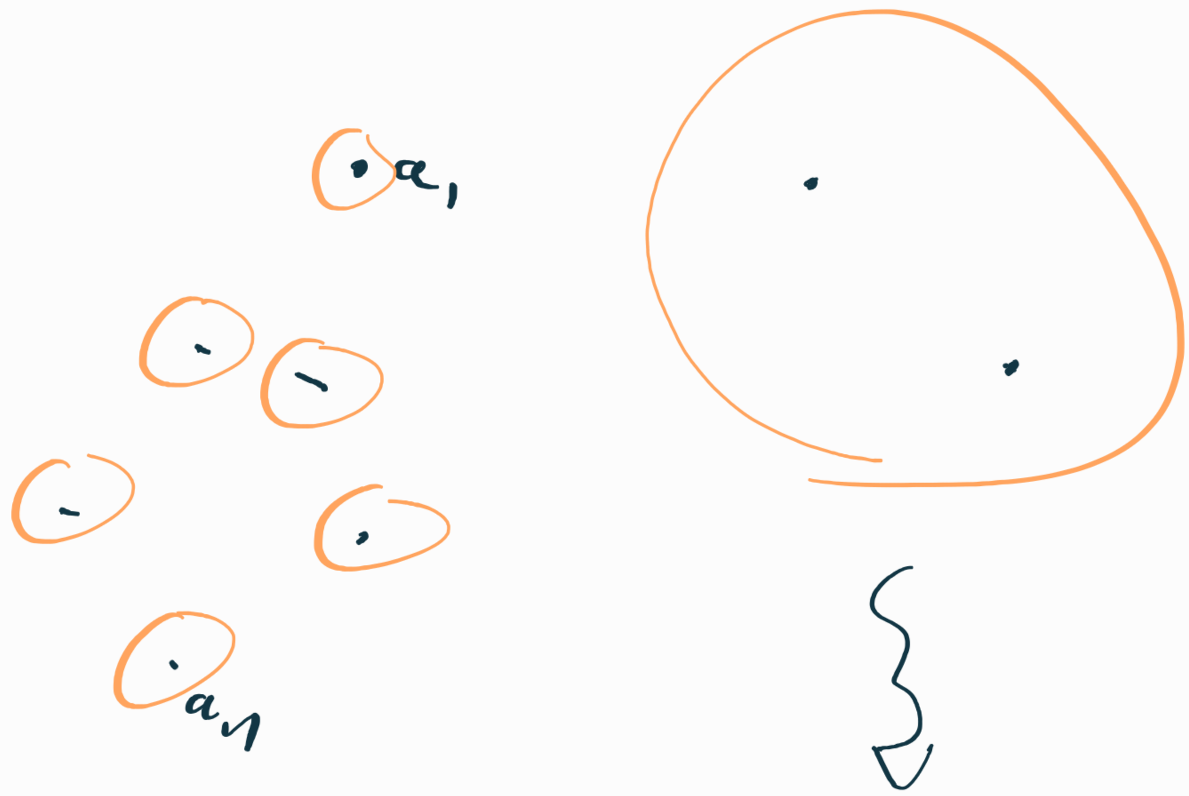
$$q(z) = p(z)(z-a)^m$$

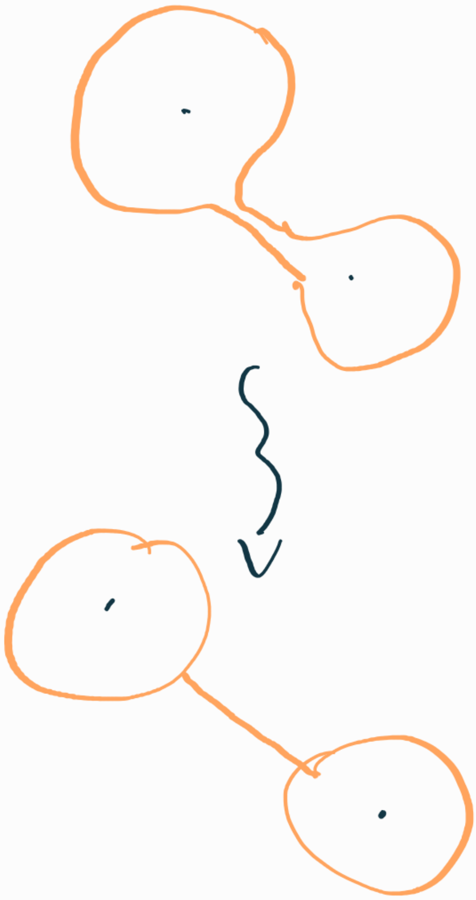
does not vanish at a,

so the homotopy

$$\gamma_t(z) = \frac{P(ta+(1-t)z)(z-a)^m}{|P(ta+(1-t)z)(z-a)^m|}$$

is well defined and witnesses the fact that the winding number of q is the same as the winding number of $(z-a)^m$ if q has no other roots. In the general case





$\square$

Since the fiberwise maps T are the restrictions of a bundle map to each fiber, T varies continuously as we move around the base, thus q_+ & q_- do as well, and the decomposition $V = V_+ \oplus V_-$ varies continuously too.

↳ So, the subspaces $V_{\pm}$ assemble (continuously) into subbundles E_+ & E_- of E, and they satisfy the desired properties because they all be check fiberwise.

∎

(Pf of ⑤).

↳ $$[E, z + b'(x)] \simeq [E_+, z + b'(x)] \oplus [E_-, z + b'(x)].$$

$$[E_+, z + b'(x)] \simeq [E_+, b'(x)] \simeq [E, id]$$

since $t \mapsto zt + b'(x)$ is a homotopy through g. clutching function.

Similarly, we get

$$[E_-, z+tb'(x)] = [E_-, z]$$

due to the $\exists$ of

$$t \mapsto z + tb'(x). \qquad \blacksquare$$

Pf of injectivity of (map in) FPT.

We'll actually define

$$\nu : K(X \times S^2) \longrightarrow K(X) \otimes \mathbb{Z}[H]/(H-1)^2$$

and verify $\nu\mu = \mathrm{id}$.

$$\nu([E, z^{-m} q]) := ((n+1)E)_- \otimes (H-1) + E \otimes H^{-m}$$

Dep. on n

Independent of n by a general property of $(\ )_-$, since.

$$((n+2)E)_- \cong ((n+1)E)_-$$

In turn we can establish:

$$[(n+2)E, L^{n+1}_q] \cong [(n+1)E, L^n_q] \oplus [E, id].$$

For $[E, id]$, the E_--part of E is zero because the corresponding linear clutching function is $z + 0 = b(x)$, and no e.v. of b lie outside the unit disk in $\mathbb{C}$.

Dep on n. By another calc. —

compare $\nu([E, z^{-n}q]) \overset{?}{=} \nu([E, z^{-n-1}(zq)])$.

↳ Straightforward ex. $(H-1)^2 = 0$ $\Rightarrow$ $H(H-1) = H-1$

Dep on t_0. $\to$ Have htpys. through. Laurent. poly. clutching func.

$$\nu_m(E\otimes H^{-m}) = \nu([E, z^{-m}])$$

$$= E_- \otimes (H-1) + E \otimes H^{-m}$$

$$= E \otimes H^{-m}.$$

∎