Hopf invariant one (hooray!)

Theorem The following are true exactly when $n \in \{1,2,4,8\}$:

① $\mathbb{R}^n$ is division algebra,

② TS^{n-1} is trivial. "S^{n-1} is parallelizable"

$$\begin{array}{c} TS^{n-1} \\ \downarrow \\ S^{n-1} \end{array} \cong S^{n-1} \times \mathbb{R}^n.$$

Def. A division algebra structure on $\mathbb{R}^n$ is a $\mu : \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}^n$ such that $\mu(x \otimes y) = 0 \Rightarrow x = 0$ or $y = 0$.

Prop. If $\mathbb{R}^n$ has a div. algebra structure then $\mathbb{R}^n$ has a unital one.

Pf. Fix any $e \in \mathbb{R}^n$ of unit length, and pick an iso. $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\phi(e^2) = e$. Post-compose our multiplication with ϕ so that we may assume that $e^2 = e$. Denote by α the map $\mathbb{R}^n \to \mathbb{R}^n$ defined by $\alpha : x \mapsto \mu(e \otimes x)$, and likewise $\beta : x \mapsto \mu(x \otimes e)$.

What is $\ker \alpha$? If $\alpha(x) = \mu(e \otimes x)$ then $x = 0$.

$\ker \alpha = \{0\}$.

Now precompose μ with $\alpha^{-1} \otimes \beta^{-1}$; the resulting map now satisfies

$$e \circ x = \mu(\alpha^{-1}(e) \cdot \beta^{-1}(x))$$

$$= \mu(e \cdot \beta^{-1}(x))$$

$$= x.$$

Likewise on the other side... $\blacksquare$

<u>Defn.</u> An H-space structure on S^{n-1} is the data of a continuous map

Hopf.

$$S^{n-1} \times S^{n-1} \longrightarrow S^{n-1}$$

which is unital. ← (i.e. $\exists$ a double-sided identity)

<u>Ex.</u> We have the division algebras $\mathbb{R}$, $\underset{\mathbb{R}^2}{\mathbb{C}}$, $\underset{\mathbb{R}^4}{\mathbb{H}}$, $\underset{\mathbb{R}^8}{\mathbb{O}}$.

Cayley-Dickson construction.

↳ These give rise to H-space structures by $(x,y) \mapsto \dfrac{xy}{|xy|}$

Prop Any division algebra structure on $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^4$, or $\mathbb{R}^8$ is isomophic to $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively.

$\hookrightarrow$ So in fact, this is an exhaustive list of examples.

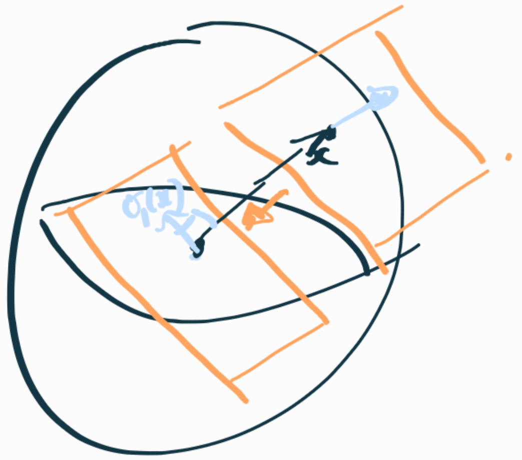
Prop. If either $\mathbb{R}^n$ has a division algebra structure or S^{n-1} is parallelizable, then S^{n-1} has a H-space structure.

Pf. We've handled the division algebra case already.

Now suppose TS^{n-1} is trivial. There then exist everywhere linearly-indep. sections $\sigma_1, \dots \sigma_{n-1}$ of TS^{n-1}.

$$\begin{array}{c} TS^{n-1} \\ \downarrow \\ S^{n-1} \end{array}$$

$(x, \sigma_1(x), \dots, \sigma_{n-1}(x))$ is a tuple of n vectors in $\mathbb{R}^n$, all linearly indep.



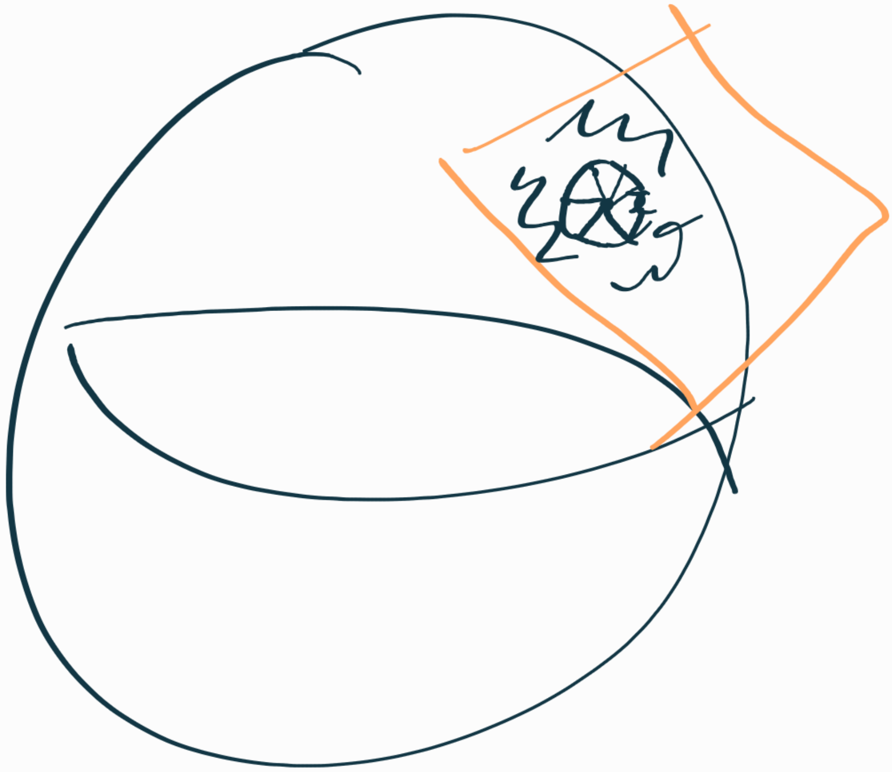
$S^{n-1} \subset \mathbb{R}^n$

We can apply Gram-Schmidt to the family $(x, \sigma_1(x), \dots, \sigma_{n-1}(x))$, to obtain an orthonormal family.

By possibly swapping two of the σ_is we can arrange that there is a $\alpha \in SO(n)$ taking $\alpha e_{i+1} = \sigma_i(e_1)$.

$$\Big(e_1, \underset{\overset{\|}{e_2}}{\sigma_1(e_1)}, \dots, \underset{\overset{\|}{e_n}}{\sigma_{n-1}(e_1)}\Big). \qquad (\#)$$

SO(n) is path-connected, so...



we can continuous perturb our family $\sigma_1, \dots, \sigma_{n-1}$ so that (#) is satisfied.

Assuming all of this, for each $x \in S^{n-1}$ there is a unique $\alpha_x \in SO(n)$ which takes the standard basis

$$(e_1, \dots, e_n) \text{ to }$$

$$(x, \sigma_1(x), \dots, \sigma_{n-1}(x)).$$

Define an H-space multiplication on S^{n-1} by

$$x \cdot y = \alpha_x y.$$

↳ This is unital with unit e_1, by construction. $\blacksquare$

<u>Observations for future computations.</u>

$\mathbb{Z}\langle (H-1)^{*\frac{n}{2}} \rangle$

① $\tilde{K}(S^n) = \mathbb{Z}$ when n is even

$= 0$ when n is odd.

(by Bott periodicity).

$$\begin{array}{ccc} \tilde{K}(S^2) \otimes \cdots \otimes \tilde{K}(S^2) & \longrightarrow & \tilde{K}(S^{2n}) \\ \mathbb{Z}(H-1) \cdots\cdots \mathbb{Z}(H-1) & & \end{array}$$

$$(H-1) \otimes \cdots \otimes (H-1) \longmapsto (H-1) * (H-1) * \cdots * (H-1).$$

(2) For any cpt. Hausdorff X external products give isos.

$$\tilde{K}(S^{2n}) \otimes \tilde{K}(X) \longrightarrow \tilde{K}(S^{2n} \wedge X),$$

& $$K(S^{2n}) \otimes K(X) \longrightarrow K(S^{2n} \times X).$$

(3) If we take $X = S^{2n}$, then we have an iso, given by external product.

$$\begin{array}{ccccc} K(S^{2n}) & \otimes & K(S^{2n}) & \xrightarrow{\cong} & K(S^{2n} \times S^{2n}) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z}[\alpha]/(\alpha^2) & \otimes & \mathbb{Z}[\beta]/(\beta^2) & & \mathbb{Z}[\alpha,\beta]/(\alpha^2,\beta^2) \end{array}$$

α $\quad$ β $\quad$ $\alpha\beta$

"α" = "$(H-1)^{*n}$"

In particular each element of $K(S^{2n} \times S^{2n})$ is of the form $a\alpha + b\beta + c\alpha\beta + d$.

Suppose $n-1 = 2k$ (is even), and let

$$\mu : S^{2k} \times S^{2k} \longrightarrow S^{2k}$$ define

an H-space structure. We

$$\begin{array}{ccc} K(S^{2k}) & \xrightarrow{\mu^*} & K(S^{2k} \times S^{2k}). \\ & \xleftarrow{\iota_1^*} & \\ \cong & \xleftarrow{\iota_2^*} & \cong \\ \mathbb{Z}[\gamma]/(\gamma^2) & & \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2). \end{array}$$

Maps $S^{2k} \longrightarrow S^{2k} \times S^{2k}$.

$$\boxed{\iota_1 : x \longmapsto (x, e)}$$

$$\iota_2 : x \longmapsto (e, x)$$

induce maps the other way

$$\begin{pmatrix} \mu \circ \iota_1 = id \\ \mu \circ \iota_2 = id \end{pmatrix}.$$

Ex What is $\iota_1^*(\alpha) = \gamma$

$\iota_1^*(\beta) = 0.$

Likewise. $\iota_2^*(\alpha) = 0 \qquad \iota_2^*(\beta) = \gamma.$

We know

$$\mu^*(\gamma) = a\alpha + b\beta + c\alpha\beta + d$$

$= 0$ (dim) $0.$

$$\gamma = \iota_1^*\mu^*(\gamma) = \iota_1^*(\qquad\qquad)$$
$$= a\gamma$$

$a = 1$, Completely analogously, $b = 1$.

We know:

$$0 = \mu^*(\gamma^2) = \mu^*(\gamma)^2 = (\alpha + \beta + c\alpha\beta)^2$$
$$= 2\alpha\beta$$

This is a contradiction.

This completely handles the "even" case of S^{2n}.

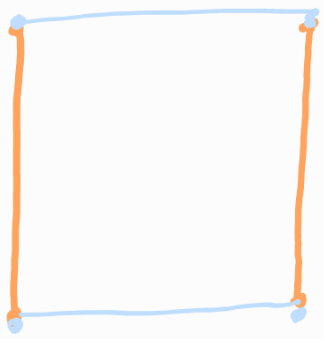
The "odd" case (of S^{2n-1}) is much much more difficult.

Let's begin with a cts.

$$g : \underbrace{S^{n-1} \times S^{n-1}} \to S^{n-1}$$

Then observe that we can write.

$$\begin{aligned} S^{2n-1} &\cong \partial(D^{2n}) \\ &= \partial(D^n \times D^n) \\ &= \underbrace{(\partial D^n \times D^n)} \cup \underbrace{(D^n \times \partial D^n)} \end{aligned}$$

$S^1 \cong$ 

Ex. What is this decomp. for $S^{2n-1} = S^3$?

Ex. What is (in general) the intersection $(\partial D^n \times D^n) \cap (D^n \times \partial D^n)$?

$$\underline{(\partial D^n \times D^n) \cap (D^n \times \partial D^n)} = S^{n-1} \times S^{n-1}$$

We can define an extension of g to all of S^{2n-1} by on $\partial D^n \times D^n$ setting

$$\hat{g}(x,y) = |x|\, g\left(\frac{x}{|x|}, y\right) \in D^n_+$$

and likewise on $D^n \times \partial D^n$ by

$$\hat{g}(x,y) = |y|\, g\left(x, \frac{y}{|y|}\right), \in D^n_-.$$

These defn. glue. since on $S^{n-1} \times S^{n-1}$ in each case.

$$\hat{g}(x,y) = g(x,y).$$

(Write. $S^n = D^n_+ \cup D^n_-$,

).

The end result is a function

$$\hat{g}: S^{2n-1} \longrightarrow S^n.$$

We care about the $n = 2k$ case.

Our recipies now form an H-space structure $\mu : S^{2k-1}\times S^{2k-1}\to S^{2k-1}$ into $\hat{\mu} : S^{4n-1}\longrightarrow S^{2n}$.

We can build a new space using $\hat{\mu}$ called the cone on $\hat{\mu}$.

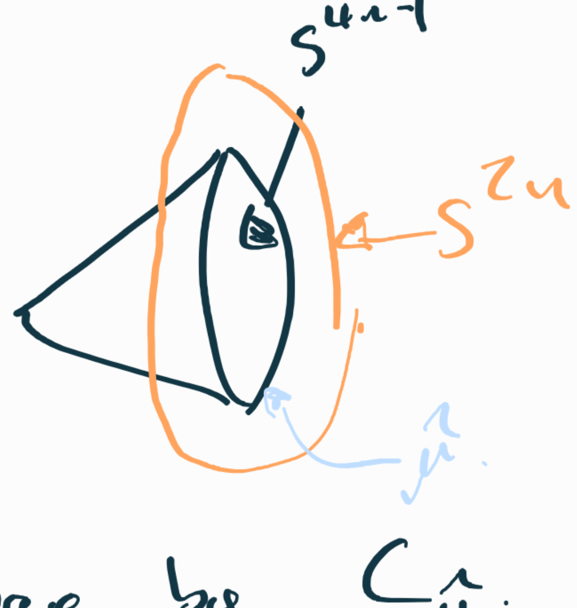
Attach a D^{4n} to the target S^{2n} of $\hat{\mu}$ via $\hat{\mu}$.

$\left(\text{i.e. } S^{2n}\sqcup D^{4n} \big/ x\sim\hat{\mu}(y)\right)$, $x \in S^{2n}$, $y \in S^{4n-1}\subset D^{4n}$

Form CS^{4n-1} ($= D^{4n}$) and then identify

$$(1,x)\sim\hat{\mu}(x)$$

$(1,x)\in CS^{4n-1}$, $\hat{\mu}(x)\in S^{2n}$.



We denote this space by $C_{\hat{\mu}}$.

We have map

$$S^{2n} \hookrightarrow C_{\hat{\mu}}$$

just given by inclusion of a closed subspace, and there is a corresponding instance of the 6-term exact sequence for K-theory:

$$\begin{tikzcd}
\tilde{K}^*(S^{2n}) \arrow[rr] & & \tilde{K}^*(C_{\hat{\mu}}, S^{2n}) \arrow[dl] \\
& \tilde{K}^*(C_{\hat{\mu}}) \arrow[ul] &
\end{tikzcd}$$

Ex. $\overset{?}{\|}$ S^{4n}

This unrolls to give.

$$0 \longrightarrow \tilde{K}(S^{4n}) \hookrightarrow \tilde{K}(C_{\hat{\mu}}) \xrightarrow{\iota^*} \tilde{K}(S^{2n}) \to 0.$$

$$\tilde{K}(S^{4n}) = \mathbb{Z}\langle\alpha\rangle \qquad \tilde{K}(S^{2n}) = \mathbb{Z}\langle\beta\rangle.$$

$\alpha \longmapsto \hat{\alpha}.$

By surj. $\exists\, \tilde{\beta} \in \tilde{K}(C_{\hat{\mu}})$ s.t. $\iota^*\tilde{\beta} = \beta$.

What is β^2? We can compute this...

$$\tilde{K}(S^2) \otimes \cdots \otimes \tilde{K}(S^2) \xrightarrow{\ \sim\ } \tilde{K}(S^{2n})$$

$$H-1 \otimes \cdots \otimes H-1 \longmapsto (H-1)^{*n}$$

$$(H-1)^2 \otimes \cdots \otimes (H-1)^2 \longmapsto \left((H-1)^{*n}\right)^2$$

$\hookrightarrow$ we conclude $\beta^2 = 0$.

Thus $\iota^* \tilde{\beta}^2 = 0$.

Equivalently, $\tilde{\beta}^2 = h\,\hat{\underset{\sim}{\alpha}}$ for some $h \in \mathbb{Z}$.

<u>Defn</u>. The integer h associated to $\hat{\mu}$ (and thus μ) is the Hopf invariant of $\hat{\mu}$ (or μ).

<u>Lemma</u>. h is well-defined.

Pf. Let $\tilde{\beta}'$ be any other lift of β.
($\iota^* \tilde{\beta}' = \beta$).

Then $\iota^*(\tilde{\beta} - \tilde{\beta}') = \beta - \beta = 0$.

$\therefore \tilde{\beta} - \tilde{\beta}' \in \ker \iota^*$, ie. $\tilde{\beta} - \tilde{\beta}' = m\hat{\alpha}$.

$\iff \tilde{\beta}' = \tilde{\beta} - m\hat{\alpha}$.

So, $\underset{\sim}{(\tilde{\beta}')^2} = (\tilde{\beta} - m\hat{\alpha})^2$

$= \tilde{\beta}^2 + \cancel{m^2\hat{\alpha}^2}^{\,0} - 2m\,\underset{\sim}{\hat{\alpha}\tilde{\beta}}$.

$= h\hat{\alpha} - 2m\,\hat{\alpha}\tilde{\beta}$.

<u>Claim</u> $\hat{\alpha}\tilde{\beta} = 0$.

Pf. Well, we at least know

$$\iota^*(\tilde\alpha\tilde\beta) = \underset{\sim}{\iota^*(\tilde\alpha)}\,\underset{\sim}{\iota^*(\tilde\beta)}$$
$$= 0, \quad \text{ie.}$$

$$\tilde\alpha\tilde\beta = m'\tilde\alpha, \qquad m' \in \mathbb{Z}.$$ On the other hand.

$$\tilde\alpha\cdot(h\tilde\alpha) = \tilde\alpha\tilde\beta^2 = \frac{m'}{2}\,\tilde\alpha\tilde\beta.$$
$$\tilde\alpha\cdot(h\tilde\alpha) = 0.$$

This implies that $\tilde\alpha\tilde\beta = 0$, since $\tilde\alpha\tilde\beta$ generates (as an abelian group) an infinite cyclic subgroup of $K(\tilde{C}_\mu)$.

∎

Lemma. Every H-space structure μ has Hopf invariant ± 1. ☒

It then remains to show that the only maps $S^{4n-1} \to S^{2n}$ which have Hopf invariant ± 1 (or indeed any odd Hopf invariant at all), occur for $n = 1, 2,$ or 4.