Remark. (Dimension/rank).

$H - 1 \in K(S^2)$ "of rank 0"

Observe that we have

$$\mathrm{Vect}^{fr}(X) \xrightarrow{\mathrm{rk}} \mathbb{N}$$

$$E \longmapsto \mathrm{rk}\,\mathbb{N}$$

$\hookrightarrow$ We set:

$\tilde{K}(X) = \ker(K(X) \to K(x_0))$

connected.

$$K(X) \longrightarrow \mathbb{Z}$$

$$E - F \longmapsto \mathrm{rk}\,E - \mathrm{rk}\,F.$$

virtual.

Given an arb. $E - F$, we can always find a complement F' of F (i.e. $F \oplus F' \cong \varepsilon^n$), we then have

$$\boxed{E - F} = (E \oplus F') - (F \oplus F') = \underline{E \oplus F' - \varepsilon^n}.$$

Picking back up...

Given $\mu : S^{2n-1} \times S^{2n-1} \to S^{2n-1}$,

we had $\hat{\mu} : S^{4n-1} \longrightarrow S^{2n} = D^{2n}_+ \cup D^{2n}_-$

We had
$$
\begin{aligned}
S^{4n-1} &= \partial D^{4n} \\
&= \partial (D^{2n} \times D^{2n}) \\
&= (\partial D^{2n} \times D^{2n}) \cup (D^{2n} \times \partial D^{2n})
\end{aligned}
$$

In particular defining,

On $\partial D^{2n} \times D^{2n}$ $\quad \hat{\mu}(x,y) = |y| \, \mu\left(x, \frac{y}{|y|}\right)$ $\quad D^{2n}_+$

On $D^{2n} \times \partial D^{2n}$ $\quad \hat{\mu}(x,y) = |x| \, \mu\left(\frac{x}{|x|}, y\right)$ $\quad D^{2n}_-$

$$
\begin{array}{ccccc}
\text{gen.} & \longmapsto & \alpha, \ \beta & \longmapsto & \text{gen.}
\end{array}
$$

$$
0 \longrightarrow \tilde{K}(S^{4n}) \longrightarrow \tilde{K}(C_{\hat{\mu}}) \longrightarrow \tilde{K}(S^{2n}) \longrightarrow 0
$$

we have
$\Phi : D^{4n} = CS^{4n-1} \hookrightarrow C_f$

$\beta^2 = h_\alpha$ ← Hopf invar.

e.g. $CS^1 =$ $= D^2$.

Lemma. If μ is an H-space mult. then $\mathrm{Hopf}(\hat\mu) = \pm 1$.

$D^{4n} \cong$

Pf. Define $f = \hat\mu$. $C_f = CS^{4n-1} \cup_{\hat\mu} S^{2n}$

$$\begin{tikzcd}
\beta \otimes \beta \arrow[rr, mapsto] & & h_\alpha \\
\tilde K(C_f) \otimes \tilde K(C_f) \arrow[rr] & & \tilde K(C_f) \\
\tilde K(C_f, D^{2n}_-) \otimes \tilde K(C_f, D^{2n}_+) \arrow[u, "\cong"'] \arrow[rr] \arrow[d, "\Phi^* \otimes \Phi^*"] & & \tilde K(C_f, S^{2n}) \arrow[u] \arrow[d, equal, "\cong"', "\Phi^*"] \\
\tilde K(D^{2n} \times D^{2n}, \partial D^{2n} \times D^{2n}) \otimes \tilde K(D^{2n} \times D^{2n}, D^{2n} \times \partial D^{2n}) \arrow[rr] \arrow[d, "\cong"] & & \tilde K(D^{2n} \times D^{2n}, \partial(D^{2n} \times D^{2n})) \arrow[d, equal, "\cong"'] \\
\tilde K(D^{2n} \times \{e\}, \partial D^{2n} \times \{e\}) \otimes \tilde K(\{e\} \times D^{2n}, \{e\} \times \partial D^{2n}) & & {}
\end{tikzcd}$$

$\qquad ||\qquad\qquad\qquad ||$

$$\tilde K(S^{2n}) \otimes \tilde K(S^{2n}) \xrightarrow[\cong \text{ by Bott.}]{*} K(S^{4n})$$

$$\begin{tikzcd}
\text{gen.} & \beta \arrow[l, mapsto] \\
\tilde K(S^{2n}) & \tilde K(C_f) \arrow[l] \\
\tilde K(S^{2n}, D^{2n}_-) \arrow[u, "\cong"] \arrow[d, "\Phi^*", "\cong"'] & \tilde K(C_f, D^{2n}_-) \arrow[l] \arrow[u, "\cong"'] \arrow[d, "\Phi^*"] \\
\text{gen } \tilde K(D^{2n} \times \{e\}, \partial D^{2n} \times \{e\}) & \tilde K(D^{2n} \times D^{2n}, \partial D^{2n} \times D^{2n}) \arrow[l, "\cong"]
\end{tikzcd}$$

We conclude that $h = \pm 1$, as desired. ∎

__Theorem (Adams)__. There exists $f: S^{4n-1} \to S^{2n}$ with $\mathrm{Hopf}(f) = \pm 1$ iff $n \in \{1, 2, 4\}$.

(or any odd integer).

__Theorem__ There are ring homs. $\psi^k : K(X) \to K(X)$ $(\forall k \in \mathbb{Z})$ which satisfy:

1. $\psi^k \circ f^* = f^* \circ \psi^k \quad \forall f: X \to Y$
2. $\psi^k(L) = L^{\otimes k}$ (L a line bundle)
3. $\psi^k \circ \psi^l = \psi^{kl}$
4. $\psi^p(a) \equiv a^p \pmod p \quad \forall a \in K(X)$, p prime.

($x \equiv y \pmod p$ for $x, y \in A$ if $[x] = [y]$ in A/pA - i.e. $x - y = pz$ for $z \in A$.)

$$\psi^k(L_1 + \cdots + L_n) = \psi^k(L_1) + \cdots + \psi^k(L_n)$$
$$= L_1^{\otimes k} + \cdots + L_n^{\otimes k} \in K(X).$$

Let us write $L^k := L^{\otimes k}$ from now on.

Recall that we can take exterior powers $\Lambda^k E$, and that for a line bundle:

- $\Lambda^0 L = \varepsilon^1$
- $\Lambda^1 L = L$
- $\Lambda^k L = 0 \quad \forall\, k > 1.$

(*)

Also, $$\Lambda^k(E \oplus F) = \bigoplus_{i=0}^{k} \underbrace{\Lambda^i(E) \otimes \Lambda^{k-i}(F)}.$$

$(e_1 + f_1) \wedge \cdots \wedge (e_k + f_k)$

$e_1 \wedge f_2 \wedge e_3 \wedge \cdots \wedge f_k$

Define $\lambda_t(E) = \sum_i \Lambda^i(E) t^i \in K(X)[t]$.

Note that (*) gets us that

$$\lambda_t(E \oplus F) = \lambda_t(E)\, \lambda_t(F).$$

When $E = L_1 \oplus \cdots \oplus L_k$, this means.

$$\lambda_t(E) = \lambda_t(L_1)\cdots\lambda_t(L_k) = \prod_{i=1}^{k}(1+L_i t).$$

The coeff. of t^i in $\lambda_t(E)$ is then

$$\sigma_i(L_1, \dots, L_k)$$

elementary symm. polynomials.

$$\sigma_1(x_1, \dots, x_k) = x_1 + \cdots + x_k$$

$$\sigma_2(x_1, \dots, x_k) = x_1x_2 + x_1x_3 + \cdots + x_{k-1}x_k.$$

$$\vdots$$

$$\sigma_k(x_1, \dots, x_k) = x_1x_2\cdots x_k.$$

If we replace the classes L_i with formal variables t_i, we get a formula:

$$(\ast\ast) \quad \prod_{i=1}^{k}(1+t_i) = 1 + \sigma_1(t_1, \dots, t_k) + \cdots + \sigma_k(t_1, \dots, t_k).$$

Every symmetric polynomial in the t_is can be written uniquely as a symmetric poly. in the σ_is by the fundamental thm of symmetric polynomials, so in particular, there is a symm. poly. s_n for each n, such that

$$t_1^n + \dots + t_k^n = s_n(\sigma_1, \dots, \sigma_k).$$

These can be easily computed directly using our $(\ast\ast)$:

$$\prod_{i=0}^{n} (x+t_i) = \underline{x^n + \sigma_1 x^{n-1} + \sigma_2 \quad + \dots + \sigma_k}$$

So taking $x = -t_i$, the LHS $= 0$, and the RHS then (after moving t_i^n to the other side) gives a formula for t_i^n in terms of the lower powers. Then the sum

$$t_1^n + \cdots t_k^n$$

is just the same over the formulas we obtain for each i...

We get formulas for the Newton polys!

$$s_1 = \sigma_1, \quad s_2 = \sigma_1^2 - \sigma_2,$$

$$s_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$$

... and so on

↳ The point is the s_i's obey

$E = L_1 \oplus \cdots \oplus L_n$

$$\boxed{s_k(\Lambda^1(E), \ldots, \Lambda^n(E))} = L_1^k + \cdots + L_n^k.$$

We define at the level of iso. classes of v.bs. an operation.

$$\psi^k(E) = s_k(\Lambda^1(E), \dots, \Lambda^n(E)),$$

We'd like to prove the thm now...

This will be easy if we can appeal to:

Splitting principle: Given any v.b. $E \to X$, & cpt. Hausdorff, the exists $F(E)$ along with $p: F(E) \to E$, such that $p^*: K(E) \to K(F(E))$ is injective and $p^*(E)$ is a sum of line bundles.

Assuming this....

Pf. of existence of ψ^ks: ① Given any $f: X \to Y$, we have $f^*\Lambda^k E = \Lambda^k f^* E$, and f^* is a ring map, so commutes with any poly. in the $\Lambda^k E$s as well.

② The claim for line bundles holds by construction.

We have additivity

$$\psi^k(E_1 \oplus E_2) = \psi^k(E_1) + \psi^k(E_2)$$

since we can write

$$\begin{tikzcd} p_{E_1}^* E_1 = & L_1 \arrow[d] & \oplus \cdots \oplus & L_n \arrow[d] & , & E_2' = p_{E_1}^* E_2 \arrow[d] \\ & F(E_1) & & F(E_1) & & F(E_1) \end{tikzcd}$$

and again we get

$$\begin{tikzcd} p_{E_2'}^* E_2' = & M_1 \arrow[d] & \oplus \cdots \oplus & M_m \arrow[d] \\ & F(E_2') & & F(E_2') \end{tikzcd}$$

Define $L_i' = p_{E_2'}^* L_i$, then

we have split both E_1 and E_2 now

over the same base $F(E_2')$.
But for line bundles

$$\psi^k(L_1' \oplus \cdots \oplus L_n' \oplus M_1 \oplus \cdots \oplus M_m)$$

$$= (L_1'^k \oplus \cdots \oplus L_n'^k) \oplus (M_1^k \oplus \cdots \oplus M_m^k)$$

$$= \psi^k(L_1' \oplus \cdots \oplus L_n') \oplus \psi^k(M_1 \oplus \cdots M_m).$$

Multiplicativity is only slightly more interesting...

$$\psi^k((L_1' \oplus \cdots \oplus L_n')(M_1 \oplus \cdots \oplus M_m))$$

$$= \psi^k(L_1' \otimes M_1) + \cdots$$

$$= (L_1')^k \otimes (M_1^k) + \cdots$$

$= \cdots$ factorize

$$= \psi^k(L_1' \oplus \cdots \oplus L_n')\,\psi^k(M_1 \oplus \cdots \oplus M_m).$$

(3) $\psi^k \circ \psi^\ell = \psi^{k\ell}$ ✓ true for line bundles.

(4)
$$\psi^p(L_1 + \cdots L_n) - (L_1 + \cdots + L_n)^p$$
$$= (L_1^p + \cdots L_n^p) - (L_1 + \cdots + L_n)^p$$
$$\overset{\text{mod } p}{\equiv} 0.$$

$\blacksquare$

Let's note one more prop.

$$\begin{array}{ccc} E_1 & & E_2 \\ \downarrow & & \downarrow \\ X_1 & & X_2 \end{array}$$

$$\begin{tikzcd} & X_1 \times X_2 \arrow[dl, "pr_1"'] \arrow[dr, "p_2"] & \\ X_1 & & X_2 \end{tikzcd}$$

$$\psi^k(E_1 * E_2) = \psi^k(pr_1^* E_1 \otimes pr_2^* E_2)$$
$$= (\psi^k pr_1^* E_1) \otimes (\psi^k pr_1^* E_2)$$
$$= (pr_1^* \psi^k E_1) \otimes (pr_2^* \psi^k E_2)$$
$$= \psi^k E_1 * \psi^k E_2.$$

... and likewise for reduced version.

$L\otimes\psi^k$ commutes with $K(X)\to K(x_0)$.

<u>Prop.</u> The operation ψ^k acts on $\tilde{K}(S^{2n})$ $\cong \mathbb{Z}$ by multiplication by k^n.

<u>Pf.</u> We proceed by induction. For $n=1$ then we know $H-1\in\tilde{K}(S^2)$, and

$$\begin{aligned}\psi^k(H-1) &= \psi^k H - \psi^k 1\\ &= H^k - 1^k\\ &= (1+(H-1))^k - 1\\ &= \cancel{1} + k(H-1) + \cancel{\binom{k}{2}(H-1)^2 + \cdots} - \cancel{1}\\ &= k(H-1) \quad \checkmark\end{aligned}$$

In the general case .

$$\psi^k\Big(\overbrace{(H-1)*\cdots*(H-1)}^{n \text{ times}}\Big) \quad \text{gen of } \tilde{K}(S^{2n})$$

$$= \psi^k(H-1) * \psi^k\Big(\overbrace{(H-1)*\cdots*(H-1)}^{n-1 \text{ times}}\Big)$$

$$= k(H-1) * k^{n-1}\Big(\overbrace{(H-1)*\cdots*(H-1)}^{n-1 \text{ times}}\Big)$$

$$= k^n \cdot (H-1)^{*n}. \qquad \blacksquare$$

Pf of HI one thm.

Start with any map $f: S^{4n-1} \to S^{2n}$, with $\mathrm{Hopf}(f)$ odd. Then recall the classes $\alpha, \beta \in \tilde{K}(C_f)$ used to define $\mathrm{Hopf}(f)$. We know

$$\psi^k \alpha = k^{2n} \alpha$$

Likewise the image (under p^*) of $\psi^k \beta$ in $\tilde{K}(S^{2n})$ is $k^n p^*(\beta)$. So,

$$\psi^k \beta = k^n \beta + \mu_k \alpha.$$

This means that.

$$\psi^\ell \psi^k = \psi^\ell\left(k^n \beta + \mu_k \alpha\right)$$

$$= k^n\left(\ell^n \beta + \mu_\ell \alpha\right) + \mu_k \ell^{2n} \alpha$$

$$= \alpha\left(k^n \mu_\ell + \ell^{2n} \mu_k\right) + k^n \ell^n \beta.$$

But $\psi^\ell \psi^k = \psi^k \psi^\ell$, so

$$k^n \mu_\ell + \ell^{2n} \mu_k = \ell^n \mu_k + k^{2n} \mu_\ell$$

$$\Rightarrow \mu_\ell\left(k^n - k^{2n}\right) = \mu_k\left(\ell^n - \ell^{2n}\right).$$

Take $k = 2$ and $\ell = 3$. Then we know

$$2^n\beta+\mu_2\alpha = \psi^2(\beta) \equiv \beta^2 \mod 2.$$
$$= h\alpha$$

$$\mu_2 - h \equiv 0 \mod 2$$

h is odd, so $\boxed{\mu_2 \text{ is odd.}}$

$$\mu_3(1-2^n)2^n = \mu_2(1-3^n)3^n.$$

In particular, 2^n must divide $1-3^n$. The proof is complete since we have:

Lemma. $2^n \mid 3^n - 1$ iff $n \in \{1,2,4\}$.

$\blacksquare$

$\blacksquare$