Final paper

- 5–10 pages
- Topic your choice ← approved by me.
- You have Aug to do it.
- As soon as you can — before the start of next week, email me with a "proposal"!
- Ideas list:

Low/No AT background.

- Find another proof of BP. – compare to ours.
- Serre–Swan thm.

  "Swan" part.

- Clifford algebra. – explain the connection to what we've seen
- The noncommutative analogue. – C^*-algebras.

AT background → consult site for more.

- Characteristic classes. – Chapter 3 of VBKT
- Chaper 4 of VBKT is similarly J-homomorphism.
- Spectra – relation to K-theory.
- "Formal group laws" parametrize complex oriented cohomology theories

  K-theory, singular cohomology, ...
- Or anything else you like...

The splitting principle remains....

There are two pieces:

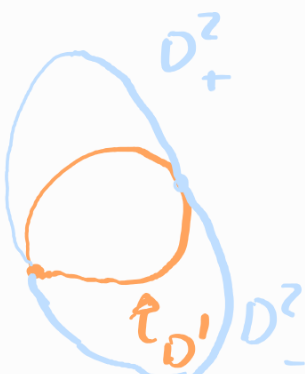
⟶ ① A calculation of $K^*(\mathbb{C}P^n)$

⟶ ② Leray–Hirsch thm for K-theory.

<u>Defn.</u> A cell complex is a topological space built by successively attaching D^n's along their boundaries S^{n-1}. (We start with a disjoint union of points.)

Eg. We can build S^2

D^2



(At each stage k, we provide the data

of a map $S^{n-1} \longrightarrow X_{k-1}$ (from the previous stage.)

This map is called the <u>attaching map</u> for the <u>cell</u> (D^n) which we are attaching.

<u>Defn</u>. A cell complex is <u>finite</u> if we started with finitely many points and attached finitely many cells.

<u>Rk</u>. A CW complex has the additional requirement that the D^n cells are attached only to cells of strictly smaller dimension.

What <u>is</u> <u>$\mathbb{CP}^n$ as a cell complex?</u>

$$\mathbb{CP}^n = \mathbb{C}^{n+1} - \{0\} \Big/ v \sim \lambda v \quad \forall v \in \mathbb{C}^{n+1} - \{0\}, \ \lambda \in \mathbb{C} - \{0\}.$$

$$= S^{2n+1} \subseteq \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$$

$$v \sim \lambda v \quad \forall v \in S^{2n+1},$$
$$\lambda \in S^1 \subset \mathbb{C}.$$

Note that each vector in S^{2n+1} with real, nonnegative last coordinate (here we view $S^{2n+1} \subset \mathbb{C}^n \times \mathbb{C}$) must be of the form

$$(w, \sqrt{1-|w|^2}) \in \mathbb{C}^n \times \mathbb{C}.$$

$(|w| \leq 1)$

The set of all such points of S^{2n+1} is the graph of $w \longmapsto \sqrt{1-|w|^2}$ defined on D^{2n}, i.e. therefore homeomorphic to D^{2n}.

Observation: every $v \in S^{2n+1}$ at all is a multiple of a vector in this graph G. If the last coordinate of v is nonzero, then v is a unique such multiple.

On the other hand, if the last coord. of v is zero then there are many possible choices of a vector in D^{2n} ...

$v = (u,0)$, $u \in \partial D^{2n} = S^{2n-1}$

So, we can alternatively build $\mathbb{C}P^n$ as the quotient of G where we identify $(u,0) \in G$ with $(u',0) \in G$ exactly when they are equal in $\mathbb{C}P^n$, in other words when $u' = \lambda u$ for $\lambda \in S^1$.

Then $\partial G = \partial D^{2n} = S^{2n-1}$ under this quotient just recovers $\mathbb{C}P^{n-1}$.

In other words, $\mathbb{C}P^n$ is built from $\mathbb{C}P^{n-1}$ by attaching a D^{2n} along the map $S^{2n-1} \cong \partial G \twoheadrightarrow \mathbb{C}P^{n-1}$.

attaching map

In this way we can pick out a $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$.

The whole "inclusion" $D^{2n} \longrightarrow \mathbb{C}P^n$ is called the characteristic map. (for the cell D^{2n}) that we are attaching.

We want to calculate $K^*(\mathbb{C}P^n)$, which we'll do in two steps.

Additive structure first:

Prop. If X is a finite cell complex with n cells, then $K^*(X)$ is a finitely generated group with at most n generators.

the finitely many points we started with count as 0-cells.

If in addition all cells of X are of even dimension, then $K^1(X)=0$, and $K^*(X)=K^0(X)$ is a free abelian group on n generators (corresponding to the cells).

It follows right away that $K^*(\mathbb{C}P^n)=\mathbb{Z}^{n+1}$.

Pf. Do induction on the number of cells; suppose that X is built from X' by attaching an n-cell (via some $S^{n-1} \to X'$).

Then we at least have an exact sequence

$$\begin{array}{c} D^n/S^{n-1} = S^n \\ \| \end{array}$$

$$\tilde{K}^*(X/X') \longrightarrow \tilde{K}^*(X) \twoheadrightarrow \tilde{K}^*(X').$$

$$y \longmapsto y'$$

By induction $\tilde{K}^*(X')$ is generated by k generators $x_1, \ldots, x_k$, and they lift to elements $\hat{x}_1, \ldots, \hat{x}_k$ of $\tilde{K}(X)$.

Given an arbitrary $y \in \tilde{K}^*(X)$, y differs from some linear combination of the $\hat{x}_i$'s by an element of the image of $\tilde{K}^*(X/X')$.

$$\tilde{K}^*(X/X') = \mathbb{Z}.$$

In particular, this image is generated by a single element $z \in \tilde{K}^*(X)$, so the set $\{z, \hat{x}_1, \ldots, \hat{x}_k\}$ (of $(k+1)$ elems) generates $\tilde{K}^*(X)$.

when all of the cells are of even dimension, then we know $K^1(S^{2n}) = 0$ $\forall n$, and so further the six-term exact sequence breaks up into

$$0 \to \tilde{K}^0(X/X') \to \tilde{K}^0(X) \leftrightarrows \tilde{K}^0(X') \to 0$$

induction.

$\cong$
$\mathbb{Z}$
$\mathbb{Z}$

by induction this is free.

by evenness

Ex Freeness in the 3rd place of a SES implies that the sequence splits.

$$\Rightarrow \quad \tilde{K}^0(X) = \mathbb{Z} \oplus \tilde{K}^0(X'),$$

so is again free.

The correspondence between generators and cells follows immediately from the fact that generators of $\tilde{K}^0(X)$ are the images of maps $\tilde{K}(X/X'') \longrightarrow \tilde{K}(X)$

$\mathbb{Z} = \tilde{K}(S^{2n})$

(X' was built from X'' by attaching a cell at some stage of building X)

It remains to compute the multiplication in $K(\mathbb{C}P^n)$:

<u>Prop.</u> There is an isomorphism

$$K(\mathbb{C}P^n) \cong \mathbb{Z}[L] \big/ (L-1)^{n+1}$$

where L corresponds to the canonical line bundle over $\mathbb{C}P^n$.

$L \downarrow \mathbb{C}P^n$ is (recall) the bundle with total space a subspace of $\mathbb{C}P^n \times \mathbb{C}^{n+1}$ according to.

$$L = \{(\ell, v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1}, v \in \ell\}.$$

<u>Proof of prop.</u>

Let's begin by noting (as we saw earlier) that $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$, and thus have an associated six-term exact sequence. We wind up with a SES

$$0 \longrightarrow K(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \to K(\mathbb{CP}^n) \to K(\mathbb{CP}^{n-1}) \to 0$$

$$K(S^{2n}) \text{``}=\text{''} \mathbb{Z}[(H-1)^{*n}] \Big/ ((H-1)^{*n})^2$$

$(*)$.

We assert using the induction hyp that

$$K(\mathbb{CP}^{n-1}) = \mathbb{Z}[L] \Big/ (L-1)^n.$$

We show at each stage that the image of $K(S^{2n})$ in $K(\mathbb{CP}^n)$ is generated by $(L-1)^n$. This will imply the claim since we'd also have $(*)$ for $(\mathbb{CP}^{n+1}, \mathbb{CP}^n)$.

View $\mathbb{CP}^n$ as the quotient of S^{2n+1} by the $S^1_{\subset \mathbb{C}}$-action. Now

$$S^{2n+1} = \partial D^{2n+2}$$
$$= \partial(D_0^2 \times D_1^2 \times \cdots \times D_n^2)$$

$$= \bigcup_{i=0}^{n} \left(D_0^2 \times \cdots \times \partial D_i^2 \times \cdots \times D_n^2 \right)$$

Now, the S^1-action on S^{2n+1} respects this decomposition, and we can let C_i for each i be the image of $D_0^2 \times \cdots \times \partial D_{\underline{i}}^2 \times \cdots \times D_n^2$ in $\mathbb{C}P^n$.

In other words, each $\underline{C_i}$ is homeomorphic to $D_0^2 \times \cdots \times \widehat{D_i^2} \times \cdots \times D_n^2$.

> Note: We talked before about $\mathbb{C}P^n$ using the notation $[z_0 : \cdots : z_n] \in \mathbb{C}P^n$, and if some z_i is nonzero, such a class is uniquely represented by
>
> $$[z_0' : \cdots : \underset{i\text{th}}{1} : \cdots : z_n'].$$

We have $\mathbb{C}P^n = \bigcup_i C_i$. Also note that $C_i \cap C_j$ is $\partial C_i \cap \partial C_j$ whenever $i \neq j$.

Let's consider C_0, and write $\partial_{\underline{i}} C_0$ for $D_1^2 \times \cdots \times \partial D_i^2 \times \cdots \times D_n^2$.

(of course, $\partial C_0 = \bigcup_i \partial_i C_0$.)

Finally

$$\begin{tikzcd}
D_i^2 \arrow[r, hook] & C_0 \arrow[r, hook] & \mathbb{C}P^n \\
\partial D_i^2 \arrow[u, hook] \arrow[r, hook] & \partial_i C_0 \arrow[u, hook] \arrow[r, hook] & C_i . \arrow[u, hook]
\end{tikzcd}$$

gen. $K(S^2) = \mathbb{Z}[H]/(H-1)^2$.

$(C_i \cong D^{2n})$

$$\begin{tikzcd}
K(D_1^2, \partial D_1^2) \otimes \cdots \otimes K(D_n^2, \partial D_n^2) \arrow[rrd, "s"] & & \\
K(C_0, \partial_1 C_0) \otimes \cdots \otimes K(C_0, \partial_n C_0) \arrow[u, "s"] \arrow[rr] & & K(C_0, \partial C_0) \\
K(\mathbb{C}P^n, C_1) \otimes \cdots \otimes K(\mathbb{C}P^n, C_n) \arrow[u] \arrow[d] \arrow[rr] & & K(\mathbb{C}P^n, \bigcup_{i \neq 0} C_i) \arrow[u, "s"] \arrow[d] \arrow[r] & K(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \arrow[ld] \\
K(\mathbb{C}P^n) \otimes \cdots \otimes K(\mathbb{C}P^n) \arrow[rr] & & K(\mathbb{C}P^n)
\end{tikzcd}$$

$x_{1,n}$

$L-1$

$C_0/\partial C_0 = \mathbb{C}P^n / \bigcup_{i \neq 0} C_i$

$\mathbb{C}P^n / \bigcup_{i \neq 0} C_i \cong \mathbb{C}P^n / \mathbb{C}P^{n-1}$

Note that we have $L-1$ "$\in$" each $K(\mathbb{C}P^n)$ in the bottom row. Denote the unique element of $K(\mathbb{C}P^n, C_i)$ mapping to $L-1$ by x_i.

Take the image of each x_i in $K(D_i^2, \partial D_i^2)$. By commutativity of the whole diagram the product $x_1 \cdots x_n \in K(\mathbb{C}P^n, \bigcup_{i \neq 0} C_i)$ is a generator.

It follows after all of this that

$$\mathrm{im}\left(K(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \longrightarrow K(\mathbb{C}P^n)\right)$$

is generated by $(L-1)^n$, and thus so too is the kernel

$$\mathrm{Ker}\left(K(\mathbb{C}P^n) \longrightarrow K(\mathbb{C}P^{n-1})\right). \quad \blacksquare$$

This completes step ①.

<u>Theorem</u>. Given a fiber bundle $F \hookrightarrow E \xrightarrow{p} B$, such that $K^*(F)$ is a free abelian group, and given classes $\{c_1, \ldots, c_n\}$ in $K^*(E)$ which restrict to generators for $K^*(F)$, then assuming a technical finiteness condition, as a $K^*(B)$-module, $K^*(E)$ is free with basis $\{c_i\}$.

$\Big(\overset{\substack{K^*(B) \\ \rotatebox{90}{$\in$}}}{x} \cdot \overset{\substack{K^*(E) \\ \rotatebox{90}{$\in$}}}{y} = p^*(x)\, y$ is the module action. $\Big)$