(Leray – Hirsch)

<u>Theorem</u>. Let $F \overset{i}{\hookrightarrow} E$ be a fiber bundle, (with F, E, B all cpt. Hausdorff).
$$\begin{array}{c} E \\ \downarrow p \\ B \end{array}$$

Suppose that $\boxed{K^*(F) \text{ is a free}}$ abelian group, and that there are elements $\{c_1, \dots, c_n\}$ of $K^*(E)$ which restrict to a set of generators of $K^*(F)$. (i.e. $\{i^*(c_1), \dots, i^*(c_n)\}$)

If either

[A] B is a finite cell complex, or

[B] F is a finite cell complex with cells only in even dimensions,

then $K^*(E)$ is free as a $K^*(B)$-module with basis $\{c_1, \dots, c_n\}$.

$$\underset{\in K^*(B)}{x} \cdot \underset{\in K^*(E)}{y} := \underset{\in K^*(E)}{p^*(x)\, y}$$

$$E \xrightarrow{p} B$$

Another way to state the conclusion is that the map

(if $E = B \times F$, $E \xrightarrow{p} B$, $F \xrightarrow{i} E$)

$$\Phi : K^*(B) \otimes K^*(F) \longrightarrow K^*(E)$$
$$c \otimes i^*(c_i) \longmapsto p^*(c)\, c_i$$

is an isomorphism.

Splitting Principle. Given a v.b. $E \to B$, with B cpt. Hausdorff, then there exists a cts. map $F(E) \xrightarrow[q]{p} B$ s.t.

1. $p^* : K^*(B) \to K^*(F(E))$ is an inj.
2. $p^* E \to F(E)$ is a sum of line bundles.

Observation: If $E \xrightarrow{p} B$ is a complex v.b. of fixed rank n, then we can form $P(E) \xrightarrow{q} B$, the projectivization

$$P(E) = E - \sigma^{-1}(B) \Big/ \sim$$

$\longrightarrow$ this recovers $\mathbb{C}P^{n-1}$ from $\begin{array}{c}\mathbb{C}^n\\ \downarrow\\ \bullet\end{array}$

We then always have $L \longrightarrow P(E)$ the canonical line bundle over $P(E)$.

$$L \subset P(E) \times E$$
$$\begin{array}{c}\cup\!\!\!\!\!\!\!\quad\\ (\ell, e)\end{array} \text{ s.t. } q(\ell) = p(e) \text{ and } e \in \ell.$$

Of course, when $\begin{array}{c}E\\ \downarrow\\ B\end{array} = \begin{array}{c}\mathbb{C}^n\\ \downarrow\\ \bullet\end{array}$, $\begin{array}{c}L\\ \downarrow\\ P(E)\end{array} = \begin{array}{c}L\\ \downarrow\\ \mathbb{C}P^n\end{array}$.

We saw last time that $1, L-1, \ldots, (L-1)^{n-1}$ was a basis for $K^*(\mathbb{C}P^{n-1})$. Then, so too is $1, L, \ldots, L^{n-1}$. It's clear that the canonical line bundle over $P(E)$ restricts over the fiber of some $b_0 \in B$ to give the canonical line bundle over $\mathbb{C}P^{n-1}$.

$$\begin{tikzcd}
L \arrow[r, hook] \arrow[d] & L \arrow[d] \\
\mathbb{C}P^{n-1} \arrow[r, hook, "i"] \arrow[d] & P(E) \arrow[d] \\
b_0 \arrow[r, hook] & B
\end{tikzcd}$$

We are now in the situation of Leray–Hirsch:

- $K^*(F) = K^*(\mathbb{C}P^{n-1})$ is free,
- the classes of $1, L, \dots, L^{n-1}$ in $K^*(P(E))$ restrict to gens,
- $\mathbb{C}P^{n-1}$ is a finite cell complex with even-dim cells.

$\hookrightarrow$ So, $K^*(P(E))$ as $K^*(B)$-module
(✡) is free with basis $\{1, L, \dots, L^{n-1}\}$.

Proof of Splitting Principle.

Form $P(E) \xrightarrow{p} B$, noting that

$$p^* : K^*(B) \longrightarrow K^*(P(E)) \text{ is}$$

injective by (✡) because 1 belongs to a basis for $K^*(P(E))$, and the module action

$$\underset{\in K^*(B)}{x} \cdot \underline{1} = p^*(x) \cdot 1 = p^*(x).$$

Now we have a picture:

$$\begin{tikzcd} p^*E \arrow[r] \arrow[d] & E \arrow[d, "q"] \\ P(E) \arrow[r, "p"] & B \end{tikzcd}$$

f.b.

$p^*E \subset P(E) \times E$ w/ elements

$$\begin{matrix} \Psi & \Psi \\ (\ell, & e) \end{matrix}$$

such that. $p(\ell) = q(e)$.

Now observe that the canonical line bundle $L \to P(E)$ is a subbundle of p^*E.

So, $p^*E \cong \underline{L \oplus E'}$, with E' obtained by fixing an inner product on E. Now do this again; another canonical line bundle splits off from the pullback of E' over the $P(E')$.

Because of how we chose the inner product on E (thus p^*E), a point of $P(E')$ is a line in a fiber of E, along with another line which is orthogonal to the first.

Fibers of the orig. bundle E have finite dim, so after finitely many steps we wind up with:

$$\begin{tikzcd}
L_1\oplus\cdots\oplus L_n \arrow[r] \arrow[d] & \cdots \arrow[r] & L\oplus E' \arrow[r] \arrow[d] & E \arrow[d] \\
\boxed{F(E)} \arrow[r] \arrow[rrr, bend right] & \cdots \arrow[r] & P(E') \arrow[r] & P(E) \arrow[r] & B
\end{tikzcd}$$

Of course, by construction $L_1\oplus\cdots\oplus L_n$ is a direct sum of line bundles and the pullback of E by $F(E)\to B$.

Each of the induced maps on K-thy are injective as we observed at the beginning so their composite is as well!

This $F(E)$ thing is the flag bundle of E. A complete flag in a v.s. V is a sequence of strictly increasing subspaces

$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$ w/ $\dim V = n$.

Likewise $\begin{array}{c} F(E) \\ \downarrow \\ B \end{array}$ has fiber above $b \in B$ the space of

complete flags in E_b.

$\hookrightarrow$ In this way we can view $F(E)$ as a quotient of the bundle of n-frames associated to $\begin{array}{c} E \\ \downarrow \\ B \end{array}$. $\mathbb{R}^n \otimes E$

On the other hand, a choice of inner prod. on E gives rise to an identification between complete flags and orthonormal n-frames. modulo signs.

So, fibers of $F(E)$ over $b \in B$ can be identified by by a choice of inner prod. with the space of n orthogonal lines in E_b.

Pf of Leray-Hirsch theorem.

<u>Note.</u> If $E = F \times B$ then the classes $\{c_i\}$ can always be chosen as the pullbacks via $E \to F$ of the classes giving a basis for $K^*(F)$. The conclusion of the thm. then says that the external prod

$$K^*(B) \otimes K^*(F) \longrightarrow K^*(E)$$

is an iso.

Let $B' \subset B$ be a subspace, set $E' := E_{B'}$. Then we have a picture $(E \xrightarrow{p} B)$ $(\ast\#)$

$$\begin{tikzcd}
\arrow[r] & K^*(B,B') \otimes K^*(F) \arrow[r] \arrow[d, "p^*"] & K^*(B) \otimes K^*(F) \arrow[r] \arrow[d, "p^*"] \arrow[dd, bend left, "\Phi"] & K^*(B') \otimes K^*(F) \arrow[r] \arrow[d, "p^*"] & {} \\
\arrow[r] & K^*(E,E') \otimes K^*(F) \arrow[r] \arrow[d] & K^*(E) \otimes K^*(F) \arrow[r] \arrow[d] & K^*(E') \otimes K^*(F) \arrow[r] \arrow[d] & {} \\
\arrow[r] & K^*(E,E') \arrow[r] & K^*(E) \arrow[r] & K^*(E') \arrow[r] & {}
\end{tikzcd}$$

defn

Prop. 2.15 VBKT

Row ③ is exact.

As are rows ① & ② since $K^*(F)$ is free.

flat

Version $\boxed{A}$: Proceed by double induction

the (finitely many) cells of B, first by their dimensions, and then by the number of them.

Base case: clear, since corresponds to B being a finite discrete set.

to build B

Induction step: Suppose that we are attaching a cell D^n to B' via an attaching map $S^{n-1} \longrightarrow B'$. We also then have the characteristic map

$$\varphi : (D^n, S^{n-1}) \longrightarrow (B, B').$$

The RH composite is an iso. by induction hyp. For the middle composite to be an iso it suffices to show that the LH composite is an iso by the

Five Lemma. If we have a picture

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, two heads] & B \arrow[r] \arrow[d, "\wr"] & C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d, "\wr"] & E \arrow[d, hook] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}
$$

exact

then $C \to C'$ is an iso.

So, draw the picture.

$$
\begin{tikzcd}
K^*(B,B') \otimes K^*(F) \arrow[r, "\varphi^* \otimes \mathrm{id}", "\sim"'] \arrow[d, "\Phi"] & K^*(D^n, S^{n-1}) \otimes K^*(F) \arrow[d, "\Phi"] \arrow[dd, bend left, "\Phi"] \\
K^*(E,E') \arrow[r, "\sim"'] & K^*(\varphi(E), \varphi(E')) \arrow[d, "\wr"] \\
 &
\end{tikzcd}
$$

φ restricts on $B - B'$ to a homeomorphism onto $D^n - \partial D^n$.

D^n is contractible

So we get a v.b. iso with $D^n \times F \to D^n$.

$K^*(D^n \times F, S^{n-1} \times F)$

To check that the module action map Φ on the LHS is an iso, by comm. it suffices to check that the rightmost composite is an iso, i.e. that
$\begin{matrix} E \\ \downarrow \\ B \end{matrix} = \begin{matrix} D^n \times F \\ \downarrow \\ D^n \end{matrix}$.

(1) Draw same picture as above but with $(B, B') = (D^n, S^{n-1})$.

$$\begin{tikzcd}
\arrow[r] & K^*(D^n, S^{n-1}) \otimes K^*(F) \arrow[r] \arrow[d, "p^*"'] \arrow[d, phantom, shift left=6ex, "\|"] & K^*(D^n) \otimes K^*(F) \arrow[r] \arrow[d, "p^*"'] \arrow[d, phantom, shift left=6ex, "\|"] & K^*(S^{n-1}) \otimes K^*(F) \arrow[r] \arrow[d, "p^*"'] \arrow[d, phantom, shift left=6ex, "\|"] & {} \\
\arrow[r] & K^*(E, E') \otimes K^*(F) \arrow[r] \arrow[d] & K^*(E) \otimes K^*(F) \arrow[r] \arrow[d] & K^*(E') \otimes K^*(F) \arrow[r] \arrow[d] & {} \\
\arrow[r] & K^*(D^n \times F, S^{n-1} \times F) \arrow[r] & K^*(D^n \times F) \arrow[r] & K^*(S^{n-1} \times F) \arrow[r] & {}
\end{tikzcd}$$

iso since $D^n \simeq \bullet$, so reduces to the 0-dim case

iso by induction hyp — dim of S^{n-1} is $<$ dim B

strictly, (B has an n-cell)

$\therefore$ LH composite is an iso by a 2^{nd} (~~1^{st}~~) app. of the five lemma.

This proves then $\boxed{A}$.

Version $\boxed{B}$: Let's first assume $E = F \times B$.

Then swap F and B.

Look at (**). The first two rows are exact sequences since they arise by e.g. tensoring $K^*(F)$ with the exact sequence

$$0 \to K^*(B,B') \to K^*(B) \to K^*(B') \to 0$$

which is split.

We again want the middle composite in (**) to be an iso. Again the RH. composite is an iso by the induction hyp. — remains to show LH is, too.

In this situation, $B/B' = S^n$ and so we can (as before) replace (B,B) with

(D^n, S^{n-1}). In this case (2nd diagram), the middle map is again an iso since $D^n \simeq *$, so LH is iso $\Leftrightarrow$ RH is.

RH being an iso is the same as asking that

$$K^*(S^{2n-1}) \otimes K^*(X) \xrightarrow[\text{prod}]{\text{external}} K^*(S^{2n-1} \times X)$$

is an iso. We've literally seen this for even dim spheres when we computed $K(S^{2n})$ (and cared about existence of H-space structures), and the same result follows immediately for odd dim spheres since K^1 of an odd sphere is K^0 of an even dim sphere.

This handles the special case $E = F \times B$.

The general case proceeds by using (**) repeatedly — we call a compact subset $C \subset B$ good if "the theorem is true for the bundle restricted to all compact subsets of $\mathbb{C}$".

The special case we just proved showed that every pt. of B has good cpt. nbhd. The base is cpt. Hausdorff, hence is covered by finitely many such sets.

It therefore is sufficient to check that unions of good sets are good.

Look at $(**)$ for

(V, V_2), and then $(V_1, V_1 \cap V_2)$

and use the five lemma.

But $V_1 \cap V_2$ is a cpt. subset of V_1, a good set.

$\boxtimes$

Thanks for coming!