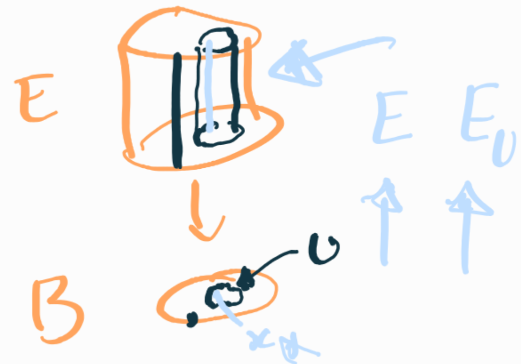
Notation.



If $E \xrightarrow{p} B$ is a v.b.

- $E_x = p^{-1}(x)$ is the fiber over x. ($x \in B$)

- $E_U = p^{-1}(U)$ is the restriction of E over U.

Defn. Let $E \xrightarrow{p} B$ be a v.b. A trivializing open cover for E is an open cover $\{U_\alpha\} \subseteq B$ for which each E_{U_α} is trivial.

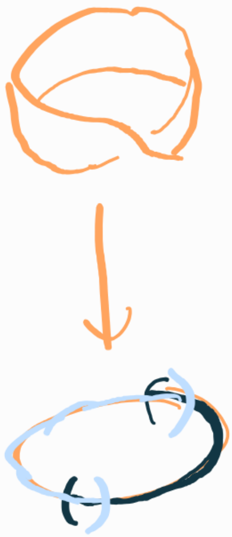
↳ i.e. there are vector bundle isomorphisms

$$E_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n \quad \text{for some } n.$$

Lemma. Every v.b. $E \to B$ has a trivializing open cover.

Pf. If $\phi_x : E_{U_x} \xrightarrow{\sim} U_x \times \mathbb{R}^{n_x}$ is a local trivialization of E around x, then $\{U_x\}$

is a trivializing open cover. ∎



<u>Construction</u>. Fix a v.b. $E \xrightarrow{p} B$, and a trivializing open cover $\{U_\alpha\}$. So we have isos.

$$\begin{tikzcd} E_{U_\alpha} \arrow[rr, "\sim"] \arrow[dr] & & U_\alpha \times \mathbb{R}^n \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$

$\phi_\alpha : E_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$.

$$\tilde{E} = \coprod_{U_\alpha} U_\alpha \times \mathbb{R}^n \Big/ \sim$$

$(x, v) \sim (x', v')$, $(x,v) \in U_\alpha \times \mathbb{R}^n$, $(x',v') \in U_\beta \times \mathbb{R}^m$

iff $\phi_\alpha^{-1}(x, v) = \phi_\beta^{-1}(x', v')$.

$$(\phi_\beta \circ \phi_\alpha^{-1})(\underset{U_\alpha}{x}, v) = (\underset{U_\beta}{x'}, v)$$

$\Longleftrightarrow$ $t_{\alpha\beta}(x)(v) = v'$ (and $x = x'$)

$$\phi_\beta \circ \phi_\alpha^{-1} : U_\alpha \times \mathbb{R}^n \longrightarrow U_\beta \times \mathbb{R}^n$$

$$\phi_\beta \circ \phi_\alpha^{-1}\big|_{U_\alpha \cap U_\beta} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

$$(x, v) \longmapsto (x, v')$$

$\Updownarrow$

$$t_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathcal{L}(\mathbb{R}^n \to \mathbb{R}^n)$$

$$x \longmapsto \left(v \mapsto (x, v) \mapsto (x, v') \mapsto v' \right)$$

$$U_\alpha \times \mathbb{R}^n \qquad U_\beta \times \mathbb{R}^n$$

<u>Defn</u>. The $t_{\alpha\beta}$s are <u>transition functions</u>.

Upshot. Given an open cover $\{U_\alpha\}$ of any topological space B, and (continuous) maps $\boxed{t_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{R}^n)}$, we should be able to build a v.b.

$(x,v) \sim (x,v) \iff t_{\alpha\alpha}(x)(v) = (v)$

$\Uparrow$

$U_\alpha \times \mathbb{R}^n$

① $t_{\alpha\alpha}(x) = id_{\mathbb{R}^n}$.

Symm. $t_{\alpha\beta}(x)(v) = v' \overset{?}{\implies} t_{\beta\alpha}(x)(v') = v$.

② $\cancel{t_{\alpha\beta}(x) = t_{\beta\alpha}(x)^{-1}}$

$\Uparrow$

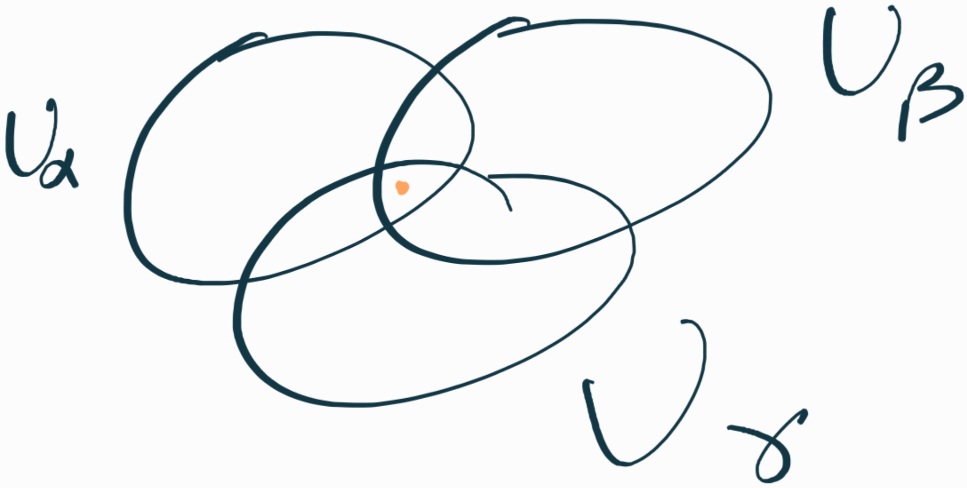
$U_\alpha \cap U_\beta$.

Trans. Cocycle condition.

$U_\alpha, U_\beta, U_\gamma$

③ $t_{\beta\gamma}(x) \circ t_{\alpha\beta}(x) = t_{\alpha\gamma}(x)$

$(x,v) \in (U_\alpha \cap U_\beta \cap U_\gamma) \times \mathbb{R}^n$



What about maps?

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & F \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

$\{U_\alpha\}$ a t.o.c. (of E & F!) of B so that:

$\phi_\alpha : E_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$, $\psi_\alpha : F_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^m$.

$$\begin{tikzcd} U_\alpha \times \mathbb{R}^n \arrow[r, "\sim", "\phi_\alpha^{-1}"'] \arrow[rrr, bend right] & E \arrow[r, "f"] & F \arrow[r, "\sim", "\psi_\alpha"'] & U_\alpha \times \mathbb{R}^m \end{tikzcd}$$

$$\Updownarrow$$

$$f_\alpha : U_\alpha \longrightarrow \mathcal{L}(\mathbb{R}^n \longrightarrow \mathbb{R}^m)$$

<u>Lemma</u>. Given v.b.s $E \searrow \swarrow F$ over B, there exists a mutually trivializing open cover.

<u>Pf</u>. Pick a t.o.c. for E, call it $\{U_\alpha\}$.

Each $x \in U_\alpha$ has an open subset $V_{\alpha,x} \subseteq B$ containing x over which F is trivial.

Now, $U_\alpha \cap V_{\alpha,x} \neq \emptyset$. So, $\{U_\alpha \cap V_{\alpha,x} : x \in U_\alpha\}$.

is the desired t.o.c.

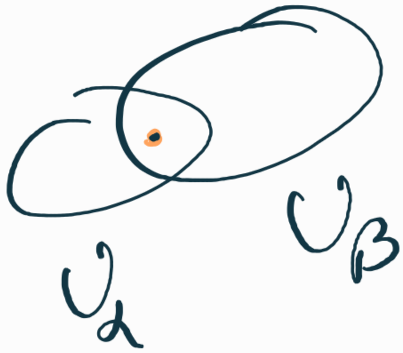
Pick t.o.c. $\{U_\alpha\}$ of E and $\{V_\beta\}$ of F. Then $\{U_\alpha \cap V_\beta\}$ is a t.o.c. of both. $\blacksquare$

$$\left.\begin{array}{l} t_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(\mathbb{R}^n) \\ s_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(\mathbb{R}^m) \end{array}\right\} \text{trans. fncts. for } E \;\&\; F \text{ respectively}$$

$$\begin{array}{ccc} \tilde{E} & \longrightarrow & \tilde{F} \\ \| & & \| \\ \left. \bigsqcup_{U_\alpha} U_\alpha \times \mathbb{R}^n \middle/ \sim \right. & & \left. \bigsqcup_{U_\alpha} U_\alpha \times \mathbb{R}^m \middle/ \sim \right. \end{array}$$

E

F



$$① \quad f_\beta \circ t_{\alpha\beta} = s_{\alpha\beta} \circ f_\alpha$$

Upshot!

Direct sum.

$$\begin{array}{ccc} E & & F \\ & \searrow \swarrow & \\ & B & \end{array} \qquad \begin{array}{c} E \oplus F \\ \downarrow \\ B \end{array}$$

$x \in$

We'd like $(E \oplus F)_x = E_x \oplus F_x$

① Pick a m.t.o.c. of E & F, call it $\{U_\alpha\}$.

$t_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{R}^n)$ $\quad s_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{R}^m)$

$$r_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(\mathbb{R}^n \oplus \mathbb{R}^m)$$

$$\begin{array}{ccc} \Downarrow & & \\ x & \longmapsto & t_{\alpha\beta}(x) \oplus s_{\alpha\beta}(x). \end{array}$$

② This defines $E \oplus F$!

e.g. now, to define

$$\iota : \underset{\sim}{E} \hookrightarrow \underset{\sim}{E \oplus F},$$

we specify for each U_α,

$$\iota_\alpha : U_\alpha \longrightarrow \mathcal{L}(\underset{\sim}{\mathbb{R}^n} \longrightarrow \underset{\sim}{\mathbb{R}^n \oplus \mathbb{R}^m})$$

$$x \longmapsto (v \longmapsto v \oplus 0).$$

$$(r_{\alpha\beta} \circ \iota_\alpha)(x)\left(\overset{\mathbb{R}^n}{\underset{v}{\in}}\right) = r_{\underset{\sim}{\alpha\beta}}(x)(x \oplus 0)$$

$$= t_{\underline{\alpha\beta}}(x)(v) \quad \checkmark$$

$$\overset{?}{=} (\iota_\beta \circ t_{\underline{\alpha\beta}})(x)(v) \quad \blacksquare$$

Tensor product. $\begin{array}{ccc} E & & F \\ & \searrow \; \swarrow & \\ & B & \end{array}$.

Then we get $E \otimes F$ by using

$$x \longmapsto t_{\alpha\beta}(x) \otimes s_{\alpha\beta}(x)$$

Hom $\begin{array}{ccc} E & & F \\ & \searrow \; \swarrow & \\ & B & \end{array}$ (by analogy $\mathcal{L}(V \to W)$)

$$\begin{array}{c} \underline{\mathrm{Hom}}(E \to F) \\ \downarrow \\ B \end{array}$$

$\underline{\mathrm{Hom}}(E \to F)_x$

$= \underline{\mathrm{Hom}}(E_x \to F_x).$

And so on... e.g. exterior powers of v.bs.

$\mathcal{L}(V \to \mathbb{R}) =: V^*$.

Defn. The dual of $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ is $\mathrm{Hom}(E \to \mathbb{R} \times B)$, denoted E^*.

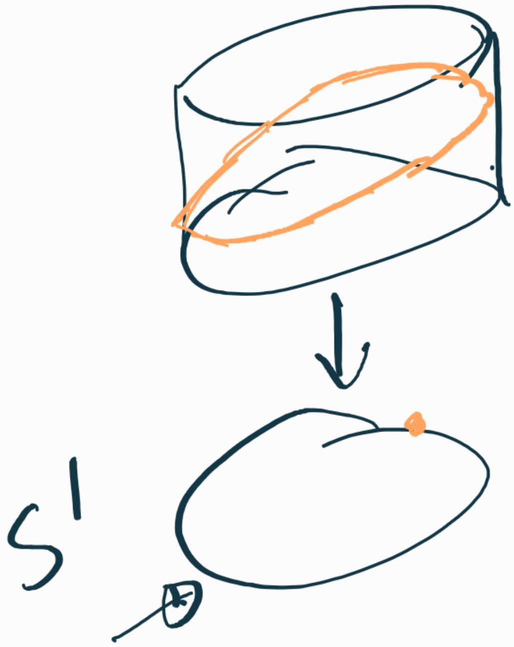
Notation. The set of all sections of a v.b. $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ is denoted $\Gamma(E \to B)$.

$(\Gamma(E))$

Claim: $\Gamma\left(\mathrm{Hom}\left(\begin{array}{c} E \\ \downarrow \\ B \end{array} \to F\right)\right) = \mathrm{Hom}(E \to F)$.

Ex.

Recall. If $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$ is a v.b. a section σ of p is a map $B \longrightarrow E$ such that $p \circ \sigma = id_B$.



Problem 4. $\begin{array}{c} M \\ \downarrow \\ S^1 \end{array}$. what is $M \oplus M$?

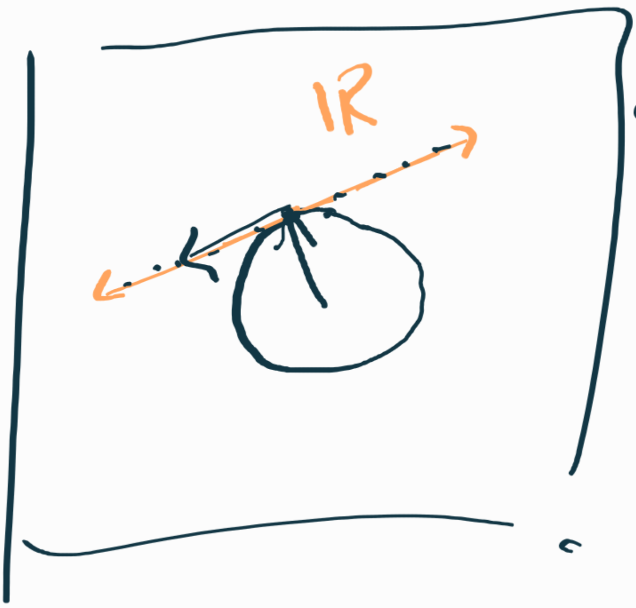
f_{00} f_{11} f_{10} f_{01} $: U_0 \cap U_1 \to GL(\mathbb{R})$

S^1

$$U_0 \times \mathbb{R}^n \cup U_1 \times \mathbb{R}^n \big/ \sim \;=\; (\underbrace{U_0 \cup U_1}_{= S^1}) \times \mathbb{R}^n$$

$M \oplus M$

TS^1 is trivial.



$S^1 \times \mathbb{R}^2$

$\cup\!\!|$

TS^1

Idea for weekend.

① Show $M \oplus M \cong \mathbb{R}^2 \times S^1$.

② Do problem 5.