Corrections - last time:

① $L \otimes L \cong B \times \mathbb{R}$

$$\begin{tikzcd} L \arrow[d] \\ B \end{tikzcd}$$

② For v.b.s, it's only the case that the kernel of a map

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr] & & F \arrow[dl] \\ & B & \end{tikzcd}$$

is a v.b. when f is of locally constant rank.

$$\begin{tikzcd} (t, x) \arrow[r, mapsto] & (t, tx) \\ \mathbb{R} \times \mathbb{R} \arrow[d, "pr_1"'] \arrow[r, "f"] & \mathbb{R} \times \mathbb{R} \arrow[d, "pr_1"] \\ t \in \mathbb{R} & \mathbb{R} \end{tikzcd}$$

Recall: $\mathrm{Vect}^k(B) = \left\{ \begin{matrix} \text{iso. classes of rank } k \\ \text{v.b.s over } B \end{matrix} \right\}$

$\mathrm{Vect}^1(B) = \{ \text{iso. classes of line bundles over } B \}$.

$\circledast$ $L_\otimes$ is a abelian group.

① Group multiplication: $\otimes$

$$[L] \cdot [L'] := [L \otimes L']$$

② Identity:

$$e := [B \times \mathbb{R}]$$

$$L \otimes (B \times \mathbb{R}) \cong L$$

Ex. Emulate this to show the same for v.b.s.

$$V \otimes \mathbb{R} \cong V$$

$$v \otimes \lambda \longmapsto \lambda v.$$

$\circledast$

$$V \otimes (U \otimes W) \cong (V \otimes U) \otimes W$$

③ Inverses:

$$[L]^{-1} := [L]$$

$L\otimes$ $\quad L\otimes L \cong B\times\mathbb{R}$

$\qquad\qquad \cong \qquad\qquad \cong \qquad \tilde{s}_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{R})$

Pick a t.o.c of B (for L).

Call it $\{U_\alpha\}$, which comes along w/

$$\varphi_\alpha : L_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}$$

$L\otimes$ + transition functions $\qquad s_{\alpha\beta} : (U_\alpha \cap U_\beta)\times\mathbb{R} \downarrow s \; (U_\alpha\cap U_\beta)\times\mathbb{R}$

$L\otimes$ $\quad L\otimes L$ is again trivial over U_α,

$\otimes$ w/ local trivialization

$$\psi_\alpha : (L\otimes L)_{U_\alpha} \xrightarrow{\sim} U_\alpha \times (\mathbb{R}\otimes\mathbb{R})$$

$$t_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \underset{\overset{\mathbb{R}}{\cong}}{(\mathbb{R}\otimes\mathbb{R})} \xrightarrow{\sim} (U_\alpha \cap U_\beta) \times \underset{\overset{\mathbb{R}}{\cong}}{(\mathbb{R}\otimes\mathbb{R})}$$

$$\mathrm{id}_{U_\alpha\cap U_\beta} \times (s_{\alpha\beta}\otimes s_{\alpha\beta})$$

$\tilde{t}_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{R}) \cong \mathbb{R}_{\neq 0}$

$\tilde{t}_{\alpha\beta} = \tilde{s}_{\alpha\beta}^{\,2}$

Lemma. A bundle with transition functions all the identity is trivial. $\boxtimes$

$\cancel{t_{\alpha\beta}} \circ \underline{f_\alpha} = \underline{f_\beta} \circ \cancel{s_{\alpha\beta}}$

Pick an inner product on L.

Now rescale the $s_{\alpha\beta}$'s to have unit length. $\Rightarrow \tilde{t}_{\alpha\beta} = 1$. Use the lemma. $\boxtimes$

④ Abelian:

$$[L] \otimes [L'] = [L'] \otimes [L]$$

$\hookrightarrow$ ① $V \otimes U \cong U \otimes V$.

Defn. A <u>fiber bundle</u> $E \xrightarrow{p} B$ <u>with fiber</u> F, is such a map p satisfying:

some topological space.

1. Each fiber $p^{-1}(x)$ is homeomorphic to F.
   $x \in B$

2. Each $x \in B$ has an open subset $U_x \subseteq B$ and a homeomorphism
   $x \in U_x$

$$\begin{tikzcd} E_{U_x} \arrow[rr, "\sim"] \arrow[dr, "p"'] & & U_x \times F \arrow[dl, "pr_1"] \\ & U_x & \end{tikzcd}$$

$$E_{U_x} = p^{-1}(U_x)$$

which respects the projection to U_x.

Assume we start w $E \to B$ a v.b.

Choose an inner product on $E \to B$.

Examples ① Sphere bundle

$$\begin{array}{c} S(E) \\ \big\downarrow \sim \\ B \end{array} \qquad S(E)_x = S(E_x), \quad x \in B$$

$$= \frac{E_x \setminus \{0\}}{e \sim te \quad \forall t \in \mathbb{R}^{>0}}$$

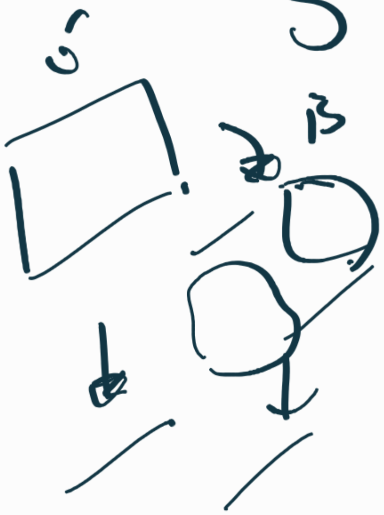
$e \in E_x$

zero section $B \to E$

$$S(E) := \frac{E \setminus \sigma(B)}{e \sim te \quad \forall e \in E,\ t \in \mathbb{R}^{>0}}$$

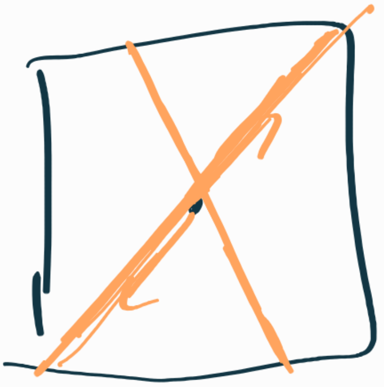
↳ $S(E)$ is a f.b. with fiber S^n where n is the rank of E

B E is constant rank.



② Projective bundle: $E \to B$.

$$P(E) := \left(E \setminus \sigma(B)\right) \Big/ e \sim te \quad \forall e \in E,\ t \in \mathbb{R}^{\neq 0}.$$



Standard fiber is $\mathbb{RP}^{n-1}$ when rank of E is n.

③ Disk bundle:

↳ Two options: $\boxed{A}$ Pick an inner product and let the fibers be honest disks.

$\boxed{B}$ Observe that D^n

$$D^2 = S^1 \times I \Big/ (\theta,0) \sim (\theta',0)$$

v.b. equivalent is

$$D(E) := S(E) \times I \Big/ \sim$$

analogous equiv. relation.

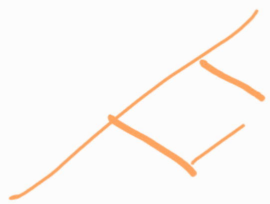
(4) Flag bundle of E

$$\begin{array}{c} F(E) \\ \big\downarrow \\ B \end{array} \qquad F(E)_{\underset{\substack{\uparrow \\ B}}{x}} = \{\text{complete flags in } E_x\}$$

Defn. Let V be a vector space. Then a complete flag in V is an strictly ascending sequence of subspaces of V

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$$

where $n = \dim V$.



⑤ Stiefel & Grassman bundles

$V_k(E)$ $\qquad\qquad$ $G_k(E)$.

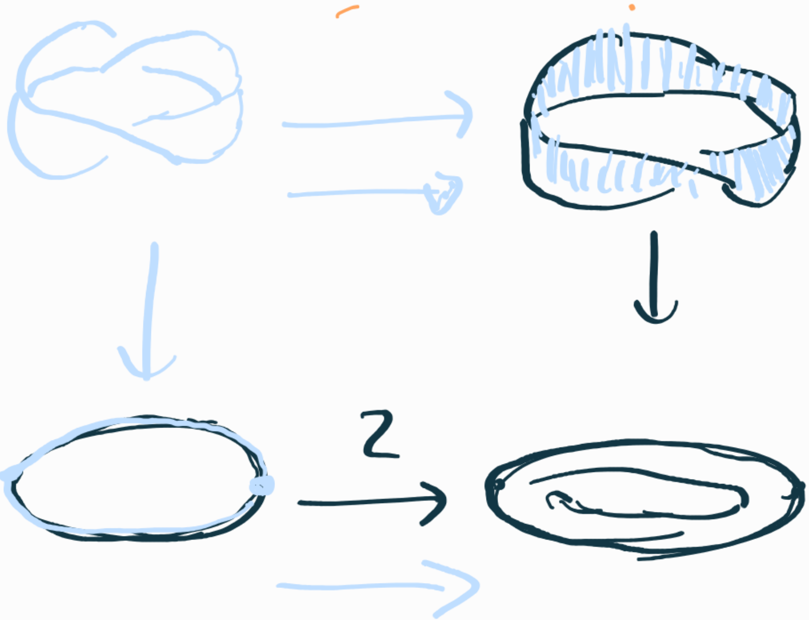
Pullback bundles.

$$
\begin{tikzcd}
(f^*E)_{f^{-1}(U_\alpha)} \arrow[d] & f^*E \arrow[r, "\tilde{f}"] \arrow[d] & E \arrow[d, "p"] & e \arrow[dl] \\
f^{-1}(U_\alpha) \arrow[r] & B' \arrow[r, "f"] & B &
\end{tikzcd}
$$

$b' \longmapsto$

$\{U_\alpha\}$

$\phi_\alpha : E_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$



Defn. Let $E \xrightarrow{p} B$ be a v.b. and let $f : B' \to B$ be a continuous map. The pullback $f^*E \to B'$ is a new v.b. with total space

$$f^*E := \{(\underset{\in E}{e}, \underset{\in B'}{b'}) : p(e) = f(b')\} \subseteq E \times B'$$

$$\begin{array}{c} f^*E \\ \downarrow \\ B' \end{array}$$

Lemma. A pullback of a trivial bundle is trivial.

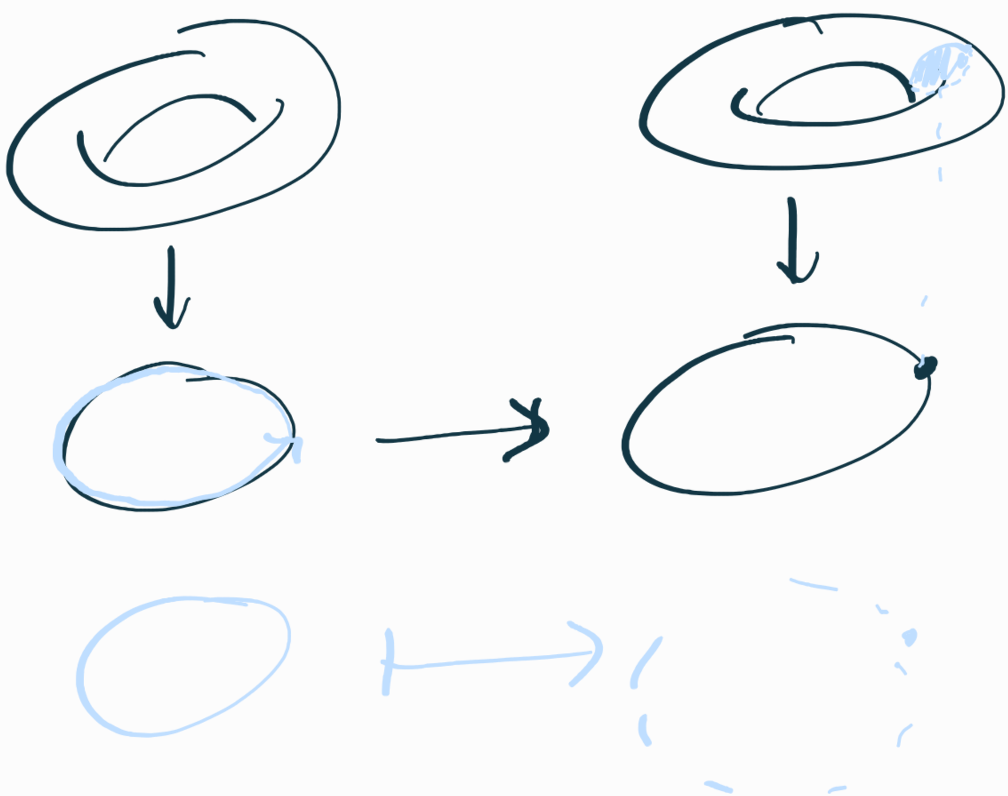
Lemma A pullback of a restriction is the restriction of a pullback.

$$f^*(B \times \mathbb{R}^n) = \{(\underbrace{b, v}_{\in B\times\mathbb{R}^n}, \underbrace{b'}_{\in B'}) : b = f(b')\} = \{(\underset{\in \mathbb{R}^n}{v}, \underset{\in B'}{b'})\} = B' \times \mathbb{R}^n \quad \square$$

Observations: ① Restrictions are pullbacks along the inclusion.

$$\begin{tikzcd} i^*E = E_{U_\alpha} \arrow[r, hook] \arrow[d] & E \arrow[d] \\ U_\alpha \arrow[r, hook, "i"] & B \end{tikzcd}$$

② When f is constant map, then f^*E is trivial.



Prop.

- $f^*(E \oplus F) \cong f^*E \oplus f^*F$.
- $f^*(E \otimes F) \cong f^*E \otimes f^*F$
- $(g \circ f)^* E \cong f^* g^* E$
- $(id)^* E \cong E$.

Prop. (Universal property of pullbacks.)

For every commutative

$$\begin{tikzcd}
F \arrow[drr, bend left, "g"] \arrow[dr, dotted, "\exists!\, \hat{g}"'] \arrow[ddr, bend right, "q"'] & & \\
& f^*E \arrow[r, dashed, "\tilde{f}"] \arrow[d, dashed] & E \arrow[d, "p"] \\
& B' \arrow[r, "f"] & B
\end{tikzcd}$$

$g = \tilde{f} \circ \hat{g}$.

We can insert the dotted maps, with the diagram still commuting.

Theorem (1.6 VBKT).

$\iff \exists\, F: I\times B' \to B$ s.t. $f_0(x) = F(0,x)$, $f_1(x) = F(1,x)$ $\forall x \in B'$.

Suppose that $E \xrightarrow{p} B$ is a v.b. and $f_0, f_1 : B' \to B$ are maps which are homotopic. Then the pullbacks $f_0^* E$ and $f_1^* E$ are isomorphic whenever B' is Hausdorff and paracompact.

If $B' = S^1$ $\quad B =$ 

what is a map $F: I \times B' \to B$

Pf. A homotopy witnessing the fact that f_0 & f_1 are homotopic is a map $F: I \times B' \to B$.

So, we can form

$$\begin{array}{c} F^*E \\ \downarrow \\ I \times B'. \end{array}$$

Prop. Given any v.b. $E \to I \times B$, then

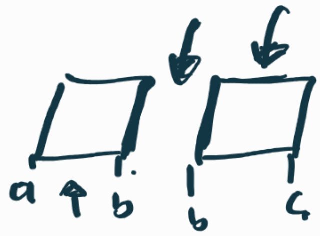
$$\begin{array}{c} E_{\{0\}\times B} \\ \downarrow \\ B \end{array} \quad \text{and} \quad \begin{array}{c} E_{\{1\}\times B} \\ \downarrow \\ B \end{array}$$

are isomorphic (where B is paracompact).

Pf. (sketch).

① If a v.b. $E \to B \times I$ is trivial over $B \times [t_0, t_1]$ and $B \times [t_1, t_2]$, then E is trivial over $B \times [t_0, t_2]$.

Ex

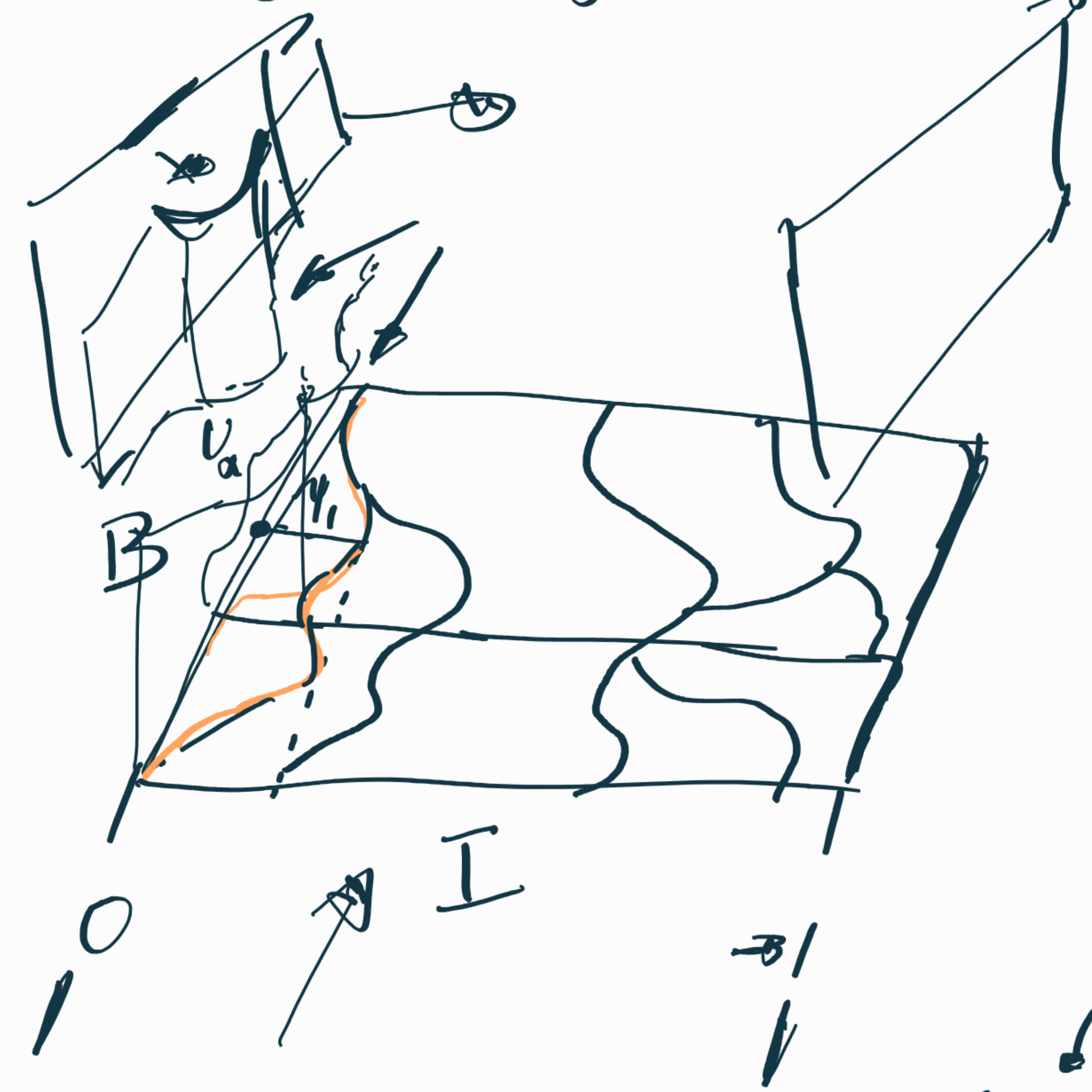


② Using ①, it is always possible to find local trivializations of $\begin{array}{c} E \\ \downarrow \\ B \times I \end{array}$ of the form $\phi : E|_{U_\alpha \times I} \xrightarrow{\sim} (U_\alpha \times I) \times \mathbb{R}^n$.

Ex.

Assuming B compact, by ② we can find a t.o.c. of F^*E of the form $\{U_\alpha \times I\}$.

by cpt. finitely many



Pick a partition of unity $\{\psi_\alpha\}$. for the $\{U_\alpha\}$.

Finitely many so number them $1, \dots, m$.

$(U_1, \dots, U_m,$
$\psi_1, \dots, \psi_m)$

m