Recall $\mathrm{Vect}^k(B) := \{$ iso classes of rank k v.b.s $\}$.

$\mathrm{Vect}^k_{\mathbb{C}}(B) := \{$ ~~~~~~~~~~~~ $\mathbb{C}$-v.b.s $\}$.

Defn Let $E \to B$ be $\mathbb{C}$-v.b. The its **conjugate** to be the new v.b. where each fiber of E is replaced by its conjugate; i.e. break $E \to B$ down into trans. fncs. $t_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{C}^n)$, and replace $t_{\alpha\beta}$ with $\overline{t_{\alpha\beta}}$.

Recall The **conjugate** $\overline{V}$ of a $\mathbb{C}$-vector space V is a new $\mathbb{C}$-v.s. where $\lambda \cdot v := \bar{\lambda} v$, and the same addition.

$\lambda \in \mathbb{C}$, $v \in \overline{V}$, $\bar{\lambda} v \in V$.

(We need B Hausdorff + paracompact.)

Prop. Let $L \to B$ be a rank 1 $\mathbb{C}$-v.b. Then $L \otimes \overline{L} \cong B \times \mathbb{C}$.

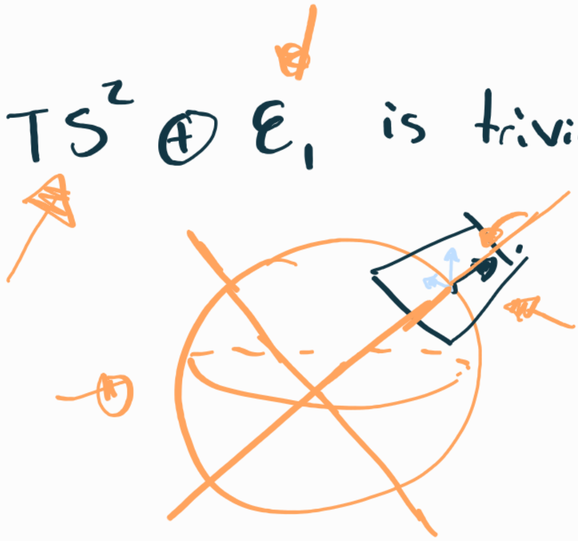
Pf. The transition funcs. are all again +-ve real numbers.

↳ Proceed as before. ∎

Let's write $\varepsilon_n \to B$ for the rank n trivial $\mathbb{C}$-v.b. over B.

Defn. A pair of v.b.s $E \to B$, $F \to B$ are **stably isomorphic** if $E \oplus \varepsilon_n \cong F \oplus \varepsilon_n$ for some $n \in \mathbb{N}$.

Fact. The direct sum $TS^2 \oplus \varepsilon_1$ is trivial.



$NS^2 \subseteq S^2 \times \mathbb{R}^3$

We also care about a slight weakening...

- Stable isomorphism defines a relation $\sim_s$
- Also, we get another relation $\sim$ by declaring $E \sim F \iff E \oplus \varepsilon_m \cong F \oplus \varepsilon_n$.

These can be different.

Ex. Show that $\sim_s$ & $\sim$ descend to equivalence relations on $\mathrm{Vect}_{\mathbb{C}}(B)$
$=$
$\{$iso. classes of $\mathbb{C}$-v.bs.$\}$. ✓

Defn. The quotient $\tilde{K}(B) := \mathrm{Vect}_{\mathbb{C}}(B) \big/ \sim$ is the reduced (complex) K-group of B.

Ex. Verify that $\tilde{K}(B)$ is a group.
cpt. Hausdorff

- $e = [\varepsilon_0]$.

- $[E]^{-1} := [E']$. $\qquad E \subseteq \varepsilon_N$

$\qquad E \oplus E' \cong \varepsilon_N$

$$[E] \oplus [E'] = [E \oplus E'] = [\varepsilon_N] = e$$

$[\varepsilon_N] \longleftrightarrow \varepsilon_N \oplus \varepsilon_0 \cong \varepsilon_0 \oplus \varepsilon_N$

i.e. $\varepsilon_N \sim \varepsilon_0$.

- Abelian grp!

What about $\mathrm{Vect}_{\mathbb{C}}(B)/\sim_s$?

- $e = [\varepsilon_0]$ ✓

$$[E] \oplus [\varepsilon_0] = [E \oplus \varepsilon_0] = [E].$$

- Commutative operation.

- But no inverses...

↳ So, $\mathrm{Vect}_{\mathbb{C}}(B)/\sim_s$ is monoid.

<u>Defn</u> The group completion $Gr(M)$ of a monoid M is: the quotient of $M \times M$ by the relation which asserts $[(m_1, m_2)] = [(m_1', m_2')]$ iff $m_1 + m_2' = m_1' + m_2$.

Identity. $[(e,e)]$ Inverses? $[(m,m')] = [(m',m)]$

↳ Since the op. is $[(m_1,m_2)] + [(m_1',m_2')] := [(m_1+m_1', m_2+m_2')]$.

Check this respects rel.

Eg. Consider $\mathbb{N} \subseteq \mathbb{Z}$.

Build $\mathbb{Z}$ as equiv. classes of pairs $(n,m) \in \mathbb{N} \times \mathbb{N}$ where we declare "$n-m$" $[(n,m)] = [(n',m')]$ whenever $n + m' = n' + m$.

Note. $Gr(M)$ comes equipped w/

$$M \longrightarrow Gr(M)$$
$$m \longmapsto \overset{m-e}{(m,e)}$$

Defn The K-group $K(B) := Gr\left(\mathrm{Vect}_{\mathbb{C}}(B) / \sim_s \right)$. (cpt Hausdorff)

Ex. Define a surjective group hom.

$$\phi : K(B) \longrightarrow \tilde{K}(B)$$
$$[(E,F)] \longmapsto [E] - [F]$$

"$E - F$"

$\mathrm{Vect}_{\mathbb{C}}(B)/\sim_s \leftarrow \mathrm{Vect}_{\mathbb{C}}(B)$

↳ implies that $\tilde{K}(B) \cong K(B) / \ker\phi$

What is the kernel of this map?

① $[(E,F)] \in \ker \phi$ whenever $E \sim_s F$.

But $E \sim_s F \Rightarrow [(E,F)] = e$.

② well, what is the image of $[(\varepsilon_n, \varepsilon_m)]$ under ϕ?

"$\varepsilon_n - \varepsilon_m$"

$\longmapsto [\varepsilon_n] - [\varepsilon_m]$

$= e$

Ex This the whole kernel....

i.e. $\ker \phi = \{[(\varepsilon_n, 0)] : n \in \mathbb{N}\} \cup \{[(0, \varepsilon_n)] : n \in \mathbb{N}\}$.

"$\varepsilon_n - 0$"

"$0 - \varepsilon_n$"

$\cong \mathbb{Z}$

We also want maps $\underset{\sim}{B'} \xrightarrow{f} \underset{\sim}{B}$ to induce maps of the corresponding K-groups...

$$\begin{array}{ccc} \underset{\sim}{\mathrm{Vect}_{\mathbb{C}}(B')} & \xleftarrow[\sim]{f^*} & \underset{\sim}{\mathrm{Vect}_{\mathbb{C}}(B)} \\ \underset{\sim}{K(B')} & \overset{K(f)}{\dashleftarrow} & \underset{\sim}{K(B)} \\ [(f^*E, f^*F)] & \longleftarrow\!\shortmid & [(E,F)] \end{array}$$

↳ **Ex.** Show this gives a well-defined group homomorphism for each possible f.

Prop. $\underset{\sim}{K(f)}$ depends only on the homotopy class of f.

Recall f and g are homotopic (equiv, belong to the same homotopy class), if $\exists F: I \times B' \to B$ s.t. $F|_{0 \times B'} = f$, $F|_{1 \times B'} = g$.

Recap. So far K is a functor to groups.

In fact $K(B)$ is a ring.

Ex. $\otimes$ of (complex) v.b.s. defines a multiplication on $\mathrm{Vect}_{\mathbb{C}}(B)$ which is compatible with $\oplus$.

i.e. the distributive law is obeyed.

In particular $\mathrm{Vect}_{\mathbb{C}}(B)/\sim_s$ is a monoid, with

$$[E]\otimes([F]\oplus[F']) = ([E]\otimes[F])\oplus([E]\otimes[F'])$$

We now get a multiplication on $K(B) = \mathrm{Gr}\left(\mathrm{Vect}_{\mathbb{C}}(B)/\sim_s\right)$ by

$$[(E,F)]\cdot[(E',F')] := [((E\otimes E')\oplus(F\otimes F'),\ (E\otimes F')\oplus(E'\otimes F))].$$

"$E-F$" "$E'-F'$"

Ex. Show this defn. respects the equiv relation defining the classes.

Also note then that e.g.

$$
\begin{aligned}
&\varepsilon^n\cdot E \quad \text{in } K(B)\\
&\quad\downarrow\\
&[(\varepsilon^n,\varepsilon^0)]\cdot[(E,\varepsilon^0)]\\
&\qquad = \\
&[(\varepsilon^n\otimes E,\varepsilon^0)].\\
&\qquad \text{"="}\\
&\varepsilon^n\otimes E,
\end{aligned}
$$

So that ε^1 (take classes) is the multiplicative unit in $K(B)$.

We get maps of rings because.

$$f^*(E\otimes F) \cong f^*E \otimes f^*F.$$

Ex. $K(\cdot) = Gr\left(\mathrm{Vect}_{\mathbb{C}}(\cdot)/\sim_s\right)$

$[(E,\varepsilon_0)]$ $E-0$ $(+)$

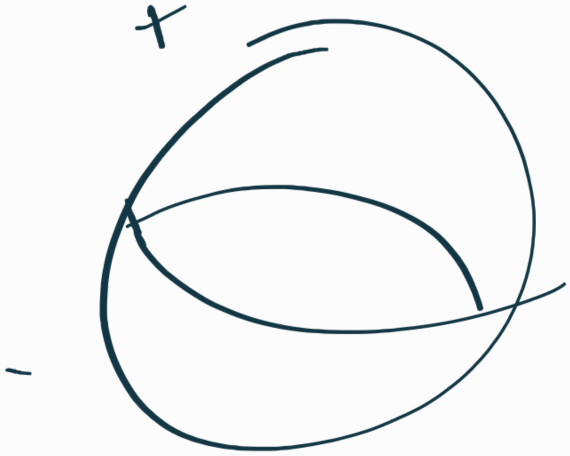
$[(\varepsilon_0, E)]$ $0-E$ $(-)$.

$V \sim_s V'$

$\Updownarrow$

$V\oplus U \cong V'\oplus U$

$\overset{?}{=} \mathbb{Z}$



$$S^n = D^n_+ \cup_{S^{n-1}} D^n_-$$

$$S^2 = D^2_+ \cup_{S^1} D^2_-$$

So given $\begin{array}{c} E \\ \downarrow \\ S^n \end{array}$, we can form

restrictions $\begin{array}{c} E \\ \downarrow \\ D^n_+ \\ z \end{array}$ $\begin{array}{c} E \\ \downarrow \\ D^n_- \\ y \end{array}$, each of which are trivial!

So, we have local trivializations...

and in particular

$$t_{-+} : \underset{y}{D^n_-} \cap \underset{z}{D^n_+} \longrightarrow GL(\mathbb{C}^n),$$

$$\begin{array}{c} \| \\ S^{n-1} \end{array}$$

It's clear that a choice of a cts. map

$$\underline{S^{n-1} \longrightarrow GL(\mathbb{C}^k)}$$

would allow us to construct a $\mathbb{C}$-v.b. over S^n.

<u>Prop.</u> The map

$$\mathrm{Top}\left(S^{n-1} \to GL(\mathbb{C}^k)\right) \to \underset{\simeq}{\mathrm{Vect}^k_{\mathbb{C}}}(S^n)$$

depends only on the homotopy class of the map $S^{n-1} \to GL(\mathbb{C}^k)$, i.e. gives a map

$$(*) \quad [S^{n-1} \to GL(\mathbb{C}^k)] \to \mathrm{Vect}^k_{\mathbb{C}}(S^n).$$

Moreover, the map $(*)$ is a bijection.

Eg. $K(S^1) \stackrel{?}{=} Gr\left(\mathrm{Vect}_{\mathbb{C}}(S^1) \big/ \sim_s \right)$

$\mathrm{Vect}^n_{\mathbb{C}}(S^1) = [S^0 \longrightarrow GL(\mathbb{C}^n)]$

$S^0 = \{\bullet_1, \bullet_2\}$

$= \mathbb{Z}.$

Ex $K(S^2) = ?$