$$K(X) := Gr\left({\mathrm{Vect}_{\mathbb{C}}(X)} / {\sim_s} \right)$$

cpt. Hausdorff.

$[E], [F) \in \mathrm{Vect}_{\mathbb{C}}(X)$

$[E] \sim [F] \iff E \oplus \varepsilon^n \cong F \oplus \varepsilon^n$

$-3 \in \mathbb{Z}$

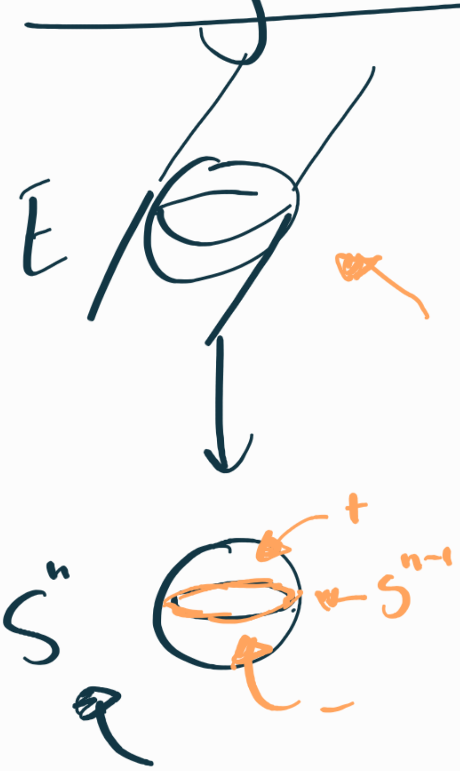
$(0,3) \in \mathbb{N} \times \mathbb{N}$

"$0-3$"

$$K(S^1) \cong \mathbb{Z}$$

$$K(S^2) \cong \; ?$$

Clutching functions



Recall that we can glue triv. bundles over open subsets of a base to build v.b. over the whole base.

The same is true for closed subsets, so long as there are finitely many.

$$S^2 = D^2_+ \cup D^2_-$$

S^1

Recall: Suppose we have a family $\{f_\alpha : U_\alpha \to Y\}$ of continuous functions defined open (or closed) subsets U_α of some space X, and moreover that the f_αs agree on the intersections of their domains.

Then $\{f_\alpha\}$ glues into a single continuous function

$$f : \bigcup_\alpha U_\alpha \longrightarrow Y.$$

(so long as $\{U_\alpha\}$ is finite.)

Ex Check this!

More formally

$$\left\{ \begin{array}{c} \text{rank } k \\ \text{v.b. over } S^n \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{maps} \\ S^{n-1} \longrightarrow GL(\mathbb{C}^n) \end{array} \right\}$$

the transition function for the closed sets D^n_+ & D^n_-.

Prop. The map

$$\left\{ \begin{array}{c} \text{maps} \\ S^{n-1} \to GL(\mathbb{C}^k) \end{array} \right\} \longrightarrow \mathrm{Vect}^k_{\mathbb{C}}(S^n)$$

descends to a bijection

$$[S^{n-1} \to GL(\mathbb{C}^k)] \longrightarrow \mathrm{Vect}^k_{\mathbb{C}}(S^n)$$

homotopy classes of maps $S^{n-1} \to GL(\mathbb{C}^k)$

Pf. Suppose that $f, f' : S^{n-1} \to GL(\mathbb{C}^k)$ give rise to isomorphic v.b. E_f & $E_{f'}$.

Let $\phi : E_f \to E_{f'}$ be an iso.

$$\begin{tikzcd}
E_f \arrow[rr] \arrow[dr] & & E_{f'} \arrow[dl] \\
& S^n & \\
D^n_+ \arrow[ur, hook] & & D^n_- \arrow[ul, hook] \\
& S^{n-1} \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}$$

$$\begin{tikzcd}
D^n_+ \times \mathbb{C}^k \cong E_f|_{D^n_+} \arrow[rr, "\phi"] \arrow[dr] & & E_{f'}|_{D^n_+} \cong D^n_+ \times \mathbb{C}^k \arrow[dl] \\
& D^n_+ &
\end{tikzcd}$$

So ϕ is equiv. to the data of a

map

$$g_+ : D^n_+ \longrightarrow GL(\mathbb{C}^k)$$

and likewise for

$$g_- : D^n_- \longrightarrow GL(\mathbb{C}^k).$$

Because ϕ is a map of v.b.s.

$$= g_+|_{S^{n-1}}$$

$\uparrow$

we want the $g_\pm$ to commute with the transition functions for E_f & $E_{f'}$.

Chill-out.

Starting with a v.b. $\begin{matrix} E \\ \downarrow \\ S^n \end{matrix}$, we have restrictions to $\underbrace{S^n}_{D^n_+ \ \& \ D^n_-}$.

These restrictions $E_+ := E_{D^n_+}$ & $E_- := E_{D^n_-}$ are non-canonically trivial!

But — we can pick trivializations

$$h_+ : E_+ \longrightarrow D^n_+ \times \mathbb{C}^k$$

$$\& \quad h_- : E_- \longrightarrow D^n_- \times \mathbb{C}^k$$

$\hookrightarrow$ the corresponding transition function is

$$f_{+-} : S^{n-1} \times \mathbb{C}^k \hookrightarrow D^n_+ \times \mathbb{C}^k \xrightarrow{h_+^{-1}} E_+|_{S^{n-1}}$$

$$\| $$

$$S^{n-1} \times \mathbb{C}^k \xleftarrow{h_-} E_-|_{S^{n-1}}$$

$\Leftrightarrow$

$$\tilde{f}_{+-} : S^{n-1} \longrightarrow GL(\mathbb{C}^k).$$

Suppose that h_+' is any other choice of local trivialization of $E_{D^n_+}$.

$$(h_+' : E_+ \longrightarrow D^n_+ \times \mathbb{C}^k).$$

We can form the composite.

$$S : D^n_+ \times \mathbb{C}^k \xrightarrow{h_+^{-1}} E_+ \xrightarrow{h_+'} D^n_+ \times \mathbb{C}^k$$

$$\Leftrightarrow \tilde{S} : D^n_+ \longrightarrow GL(\mathbb{C}^k).$$

Claim: $\tilde{S}$ is homotopic itself to a constant map.

Pf (by picture) $\tilde{S}_0 := \tilde{S}$

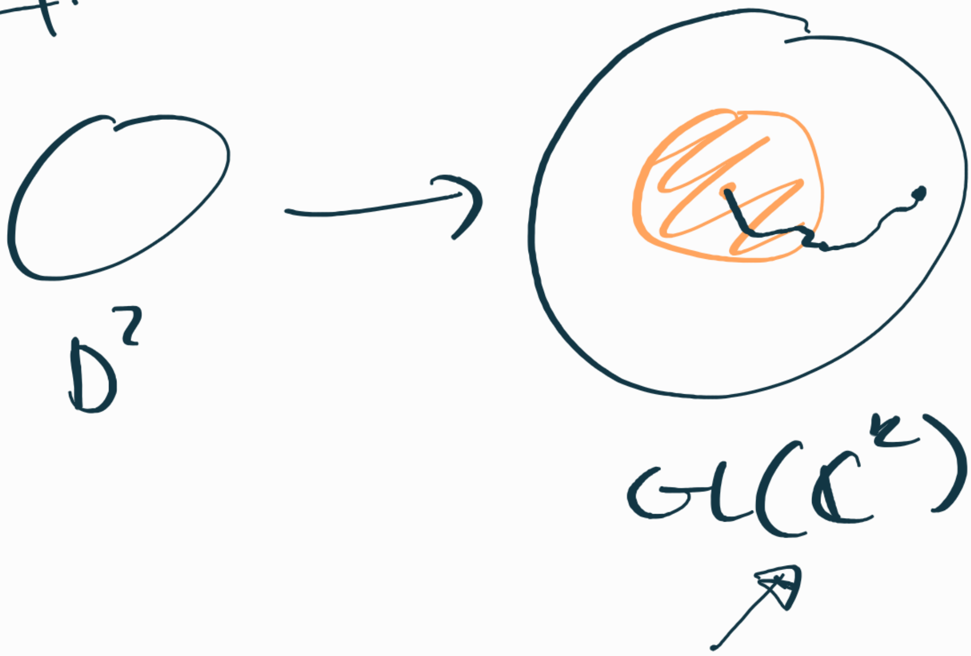
$$\longrightarrow GL(\mathbb{C}^k)$$

D^2

$\tilde{S}_t : I \times D^2 \longrightarrow GL(\mathbb{C}^k).$

Claim: Any constant $D^n \to GL(\mathbb{C}^k)$ is homotopic to the identity.

Pf.



$GL(\mathbb{C}^k)$ is path-connected!

$\hookrightarrow$ Pf (of subclaim).

We start with a matrix

$$\begin{array}{cc} \begin{bmatrix} \circ & \cdot & \cdot & \cdot \\ \circ & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} & k \\ k & \end{array} \in M_{k\times k}(\mathbb{C}).$$

and we'd like a path in $M_{n\times n}(\mathbb{C})$ to I_n, passing through invertible matrices.

$$\overset{[0,1]}{t}cR_1 + R_2 \mapsto R_2.$$

$$\begin{bmatrix} 0 & 1 & 0 & \\ 1 & 0 & 0 & \ddots \\ 0 & 0 & 1 & \\ & \vdots & & \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$t=0$ $\qquad$ $t=1.$

$$\begin{bmatrix} t & 1-t \\ 1-t & t \end{bmatrix}.$$

Final payoff: We conclude that

$$\textcircled{1}\ \tilde{S} : D^n_+ \longrightarrow GL(\mathbb{C}^k)$$

is homotopic to

$$x \longmapsto I_k .$$

This immediately gives an explicit homotopy between h_t & h'_t $\longrightarrow \textcircled{2}$

why? Recall that

$$\tilde{S} \circ h_t = h'_t ,$$

so if $\tilde{S}_t$ is s.t. $\tilde{S}_0 = \tilde{S}$ and $\tilde{S}_1 = I_k$, then

$$t \longmapsto \tilde{S}_t \circ h_t$$

gives the desired homotopy

The same is true for D^n_-, so we obtain a homotopy from $\tilde{f}_{+-}$ for (h_+, h_-) to the corresponding map for (h'_+, h'_-).

$\boxtimes$

Ex. ① Trivial bundles correspond to identity-constant maps, i.e.

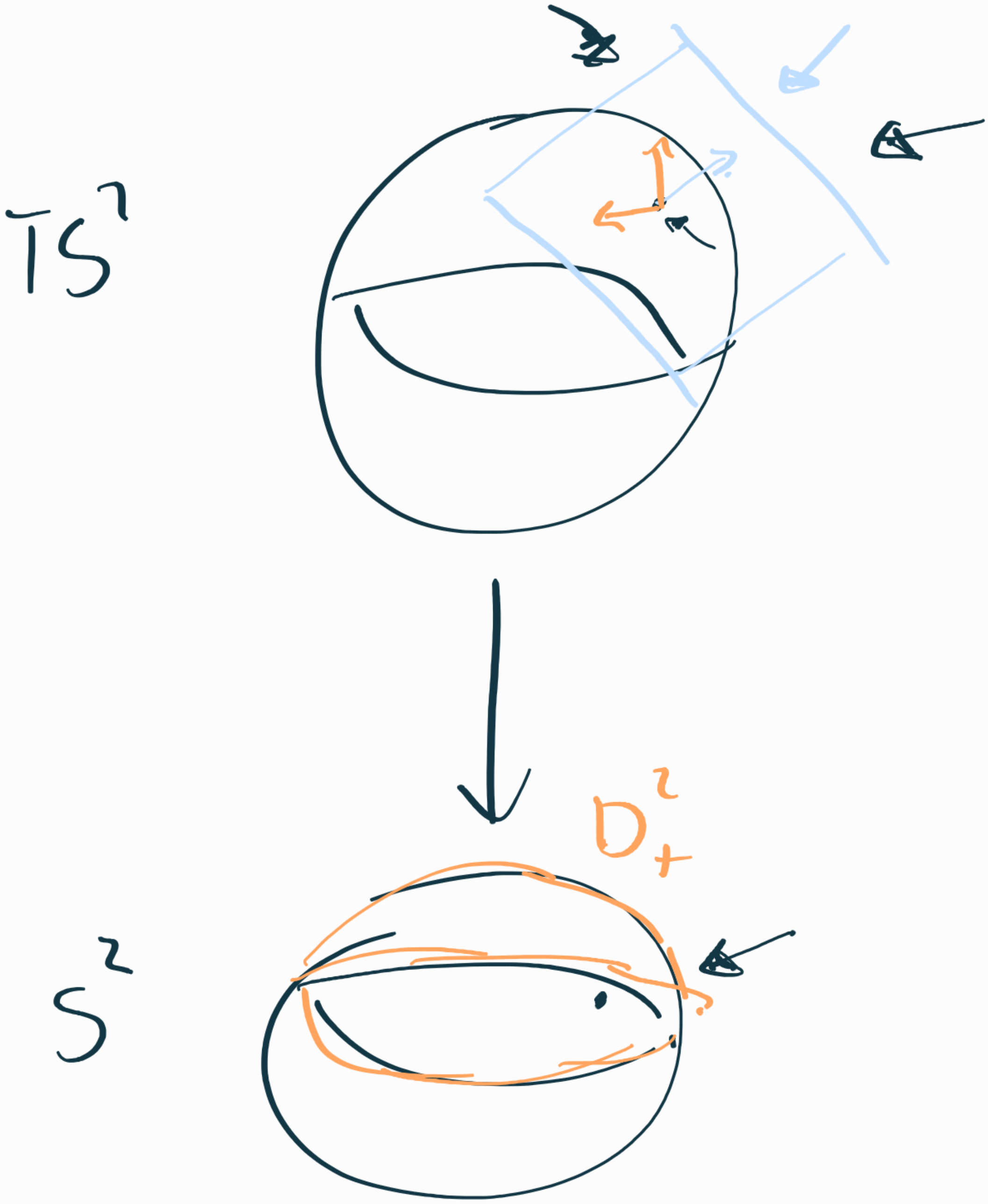
$$\begin{array}{c} S^n \times \mathbb{C}^k \\ \varepsilon_k \big\downarrow \\ S^n \end{array} \longleftrightarrow \begin{array}{c} S^{n-1} \longrightarrow GL(\mathbb{C}^k) \\ x \longmapsto I_k. \end{array}$$

② What about the example of

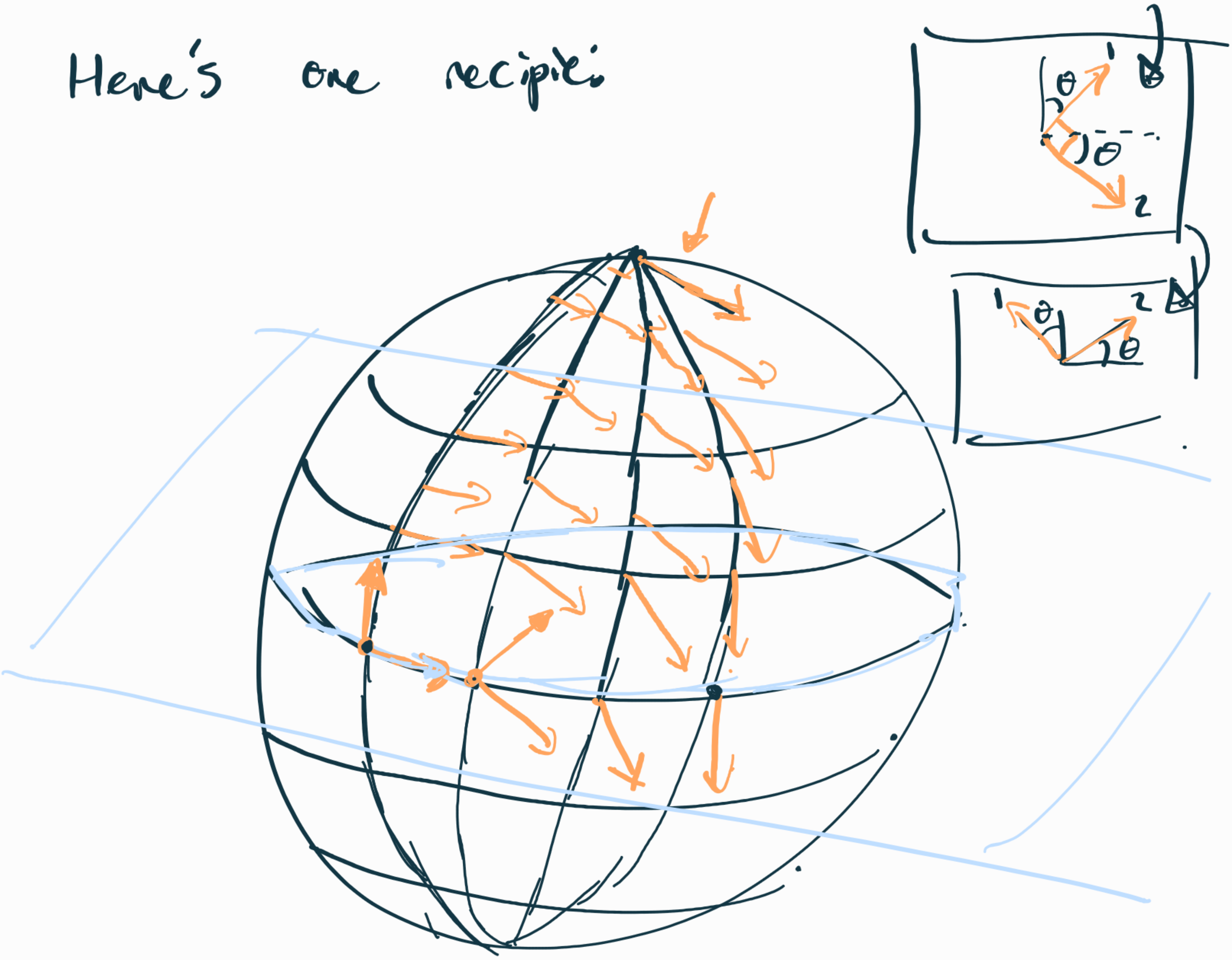
TS^2 ← except, this is a real v.b.

$$\begin{array}{c} TS^2 \\ \downarrow \ ? \\ S^2 \end{array}$$

The clutching function for this bundle is $S^1 \longrightarrow GL(\mathbb{R}^2)$.



Here's one recipe:



This picture produces for us a section $\sigma_1 : S^2 \longrightarrow TS^2$.

We get a second section $\sigma_2 : S^2 \longrightarrow TS^2$ by rotating each vector in the picture by 90°, counterclockwise, as viewed from outside the sphere.

By mirror through the equatorial plane we get a pair of linearly indep. sections of $TS^2|_{D^2_-}$ as well. $\longrightarrow$ renumbering to rotate clockwise instead.

The result is that the corresponding clutching function is

$$[0, 2\pi]/_{0\sim 2\pi} \cong S^1 \xrightarrow[2\theta \text{ rot}]{} GL(\mathbb{R}^2)$$

$$\theta \longmapsto \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

③ Canonical line bundle over $\mathbb{CP}^1$

↳ Aside. For any base B, we've seen that we can build the trivial bundle over B of any rank.

Over $\mathbb{CP}^1$, there is another:

$$\mathbb{CP}^1 = \{\text{lines in } \mathbb{C}^2\}.$$

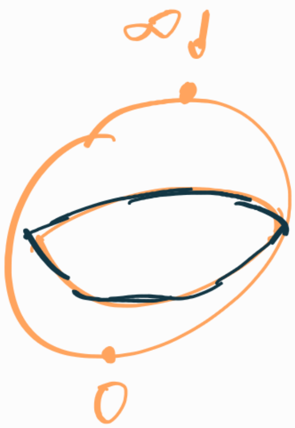
$$= \{(z_1, z_2) \in \mathbb{C}^2 - \{(0,0)\}\} \Big/ (z_1, z_2) \sim (\lambda z_1, \lambda z_2) \quad \forall \lambda \in \mathbb{C}$$

Above $[(z_1,z_2)] \in \mathbb{CP}^1$, we want the line spanned by $(z_1,z_2)\in\mathbb{C}^2$.

$$\Big\{\Big(\overset{\mathbb{CP}^1}{\overset{\Downarrow}{[(z_1,z_2)]}},\ \overset{\mathrm{span}(z_1,z_2)}{\overset{\Downarrow}{(w_1,w_2)}}\Big)\Big\} \subseteq \mathbb{CP}^1 \times \mathbb{C}^2$$

We can build local trivs. by observing:

each $[(z_1,z_2)] \in \mathbb{CP}^1$, whenever $z_1 \neq 0$, is equal to $[(1, \frac{z_2}{z_1})]$.



That means that $\mathbb{CP}^1 - \{[(0,1)]\}$ is canonically in correspondence with $\mathbb{C}$. Likewise for the other coordinate, i.e. corresponding to

The map

$$[(z_1, z_2)] \longmapsto z_1/z_2 .$$

to.

↳ The net result is that we get a bundle

$$\begin{array}{c} H \\ \downarrow \\ \mathbb{C}P^1 \\ \shortparallel \\ S^2 \end{array}$$

, called the canonical line bundle over $\mathbb{C}P^1$.

(There is an analogous construction for any $\mathbb{C}P^n$.)

$$\begin{array}{ccccc} S^1 & & & & \\ \Downarrow & & & & \\ z & \longmapsto & [(1, z)] & \longmapsto & z^{-1} \end{array}$$

This defines a map

$$\begin{aligned} S^1 &\longrightarrow GL(\mathbb{C}). \\ z &\longmapsto (z^{-1} \cdot -) \end{aligned}$$

Fundamental product formula

There is an isomorphism (explicitly expressible)

$$K(X) \otimes \frac{\mathbb{Z}[H]}{(H-1)^2} \xrightarrow{\sim} K(X \times S^2)$$

$K(S^2)$

Next time:

① Explicitly write down the map.

② Show that in $K(S^2)$, the class of H satisfies $[H]^2 + [\varepsilon_1] = [H] + [H]$.

③ Proof proper.

↳ systematically reduce to simpler cases.