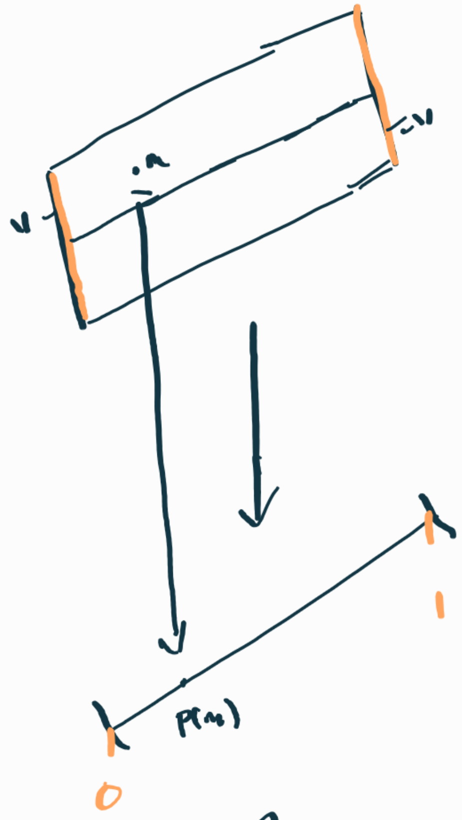
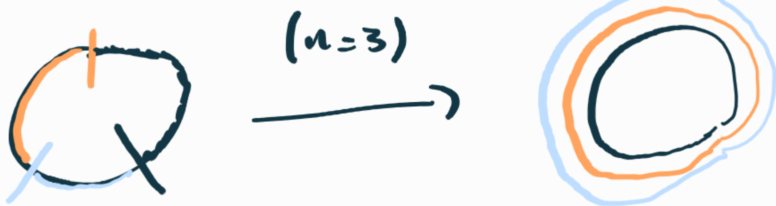
①

$$\begin{tikzcd} M \arrow[d, "p"] \\ S^1 \end{tikzcd}$$



$$\begin{tikzcd} I \times \mathbb{R} / (0,v)\sim(1,-v) \arrow[d, "p"] \\ I / 0\sim 1 \end{tikzcd}$$

$$S_1 \xrightarrow{\gamma_n} S_1$$



$$\gamma_n^* M = \{(\theta, m) : \gamma_n(\theta) = p(m)\} \subseteq S^1 \times M$$

Suppose n even.

$$\begin{tikzcd}
S^1 \times M \supseteq \gamma_n^* M \arrow[r] \arrow[dr, "pr_1"'] & S^1 \times \mathbb{R} \arrow[d, "pr_1"] \\
& S^1
\end{tikzcd}$$

(n = 4)

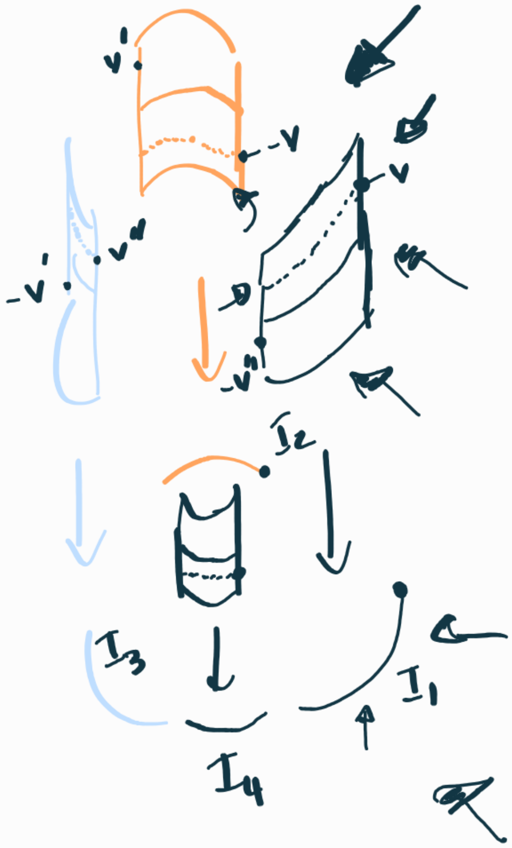
$I_i \subseteq S^1$

$S^1 \times M$

$\cup$

$I_1 \times \mathbb{R} \sqcup I_2 \times \mathbb{R} \sqcup I_3 \times \mathbb{R} \sqcup I_4 \times \mathbb{R} \to \gamma_n^* M$

$\sim$



$(-1)^n = 1 \iff n$ even.

Claim A rank 1 real v.b. L over base B is trivial iff L has a nowhere zero section.

Pf. Starting with a iso. $L \overset{\sim}{=} B \times \mathbb{R}$, we observe that this is the data of

$$\begin{array}{ccc} L & \underset{\psi}{\overset{\phi}{\rightleftarrows}} & B \times \mathbb{R} \\ \cup & & \\ L_x & & \\ \Downarrow & & \\ v & \longmapsto & (x, ?) \end{array} \qquad (x \in B)$$

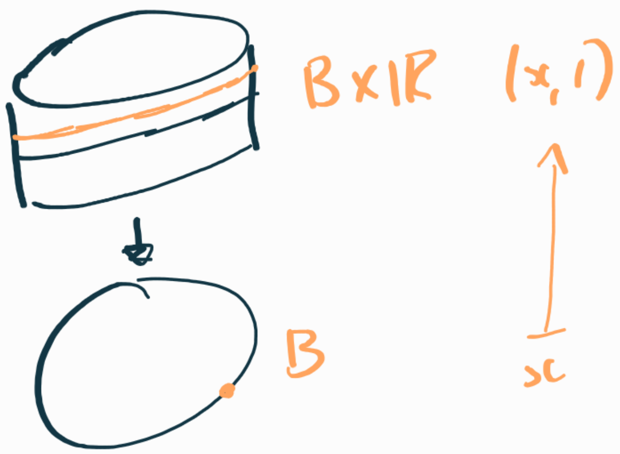
$$B \times \mathbb{R} \xrightarrow{\psi(x,\lambda)} L$$

$$\psi(x,1) : B \longrightarrow L$$

is the same as a map

$$L \longrightarrow \mathbb{R},$$

which is an iso on each fiber.



Now starting with a nowhere-zero section $\sigma : B \longrightarrow L$, we can build a map $L \longrightarrow B\times\mathbb{R}$ by

$$\begin{array}{ccc} \sigma(x)\in L_x & \longrightarrow & B\times\mathbb{R} \\ \cup & & \\ v=\lambda_v\,\sigma(x) & \longmapsto & (x,\lambda_v), \\ \uparrow & & \end{array}$$

we can check that this is an iso in local trivializations. $\boxtimes$

② $GL(\mathbb{C}^n)$ is path-connected.

i.e. $\forall\, g, g' \in GL(\mathbb{C}^n)$ $\exists$ a cts.

$$f : I \longrightarrow GL(\mathbb{C}^n)$$

s.t. $f(0) = g$ and $f(1) = g'$.

There is no path from $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$g$$

$$\begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & \vdots & & \\ 1 & \vdots & \vdots & & \\ 0 & \vdots & \vdots & \cdots & \\ \vdots & \vdots & 0 & & \\ \vdots & & 1 & & \\ & & 0 & & \\ 0 & 0 & 0 & & \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$t \longmapsto \begin{bmatrix} \\ \\ \end{bmatrix}$$

$\Downarrow$

③/④: Correction.

$\uparrow$

$$\begin{array}{c} H \\ \downarrow \\ \mathbb{C}P^1 \end{array}$$

$\mathbb{C}P^1 \cong S^2$ (use <u>stereographic</u> projection!)

$0 \in \mathbb{C}$

$\mathbb{C} \cong \mathbb{C}P^1 - \{[(1,0)]\}$

"∞" $\in \mathbb{C}$

$\mathbb{C} \cong \mathbb{C}P^1 - \{[(0,1)]\}$

$$[(z_1, z_2)] \longmapsto \frac{z_1}{z_2} \text{ or } \frac{z_2}{z_1}$$

$$e^{i\theta} \overset{\xleftarrow{z_2/z_1}}{\longmapsto} [(1, e^{i\theta})]$$

$$= [(e^{-i\theta}, 1)]$$

$$\overset{z_1/z_2}{\longmapsto} e^{-i\theta}$$

$$f(z) = (z^{-1})$$

(5)

$\widetilde{H \oplus H}$ $\qquad$ $\widetilde{(H \otimes H.)} \oplus \mathcal{E}'$

CFs:

$$z \mapsto \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} \simeq \qquad z \mapsto \begin{bmatrix} z^2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $$

Are these homotopic?

$$(z \oplus id)(id \oplus z)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \wr\!\wr$$

$$\begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad z \oplus id$$

X_t

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then our desired homotopy is $X_t^{-1} \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} X_t$

If $\bar{E}_f$ is the v.b. obtained from a clutching function f, then (by the same argument)

$$\bar{E}_f \oplus \bar{E}_g \cong \bar{E}_{f\cdot g} \oplus \varepsilon_n$$

(f, g are clutching functions for rank n v.bs)

<u>Rk</u>. If I write H for the class of $[H] \in K(S^2)$, then we've just shown $H \cdot H + 1 = H + H = 2H$ i.e. $H^2 + 1 = 2H$.

formal symbol

$$\begin{tikzcd}
\mathbb{Z}[H] \arrow[r] \arrow[d] & K(S^2) \\
\frac{\mathbb{Z}[H]}{(H-1)^2} \arrow[ur, dotted] &
\end{tikzcd}$$

$$H \longmapsto \text{"}H\text{"}$$

⑥ $Gr(\mathbb{N}_\infty) = ?$

An arb. elem. of $Gr(\mathbb{N}_\infty)$ is rep'd by

$$\begin{array}{ccc} n & - & m \\ \uparrow & & \uparrow \\ \mathbb{N}_\infty & & \mathbb{N}_\infty \end{array}$$

If $(n-0)+(\infty-0) = (\infty-0)$

$$n - 0 = 0 - 0$$

↳ $Gr(\mathbb{N}_\infty) = \{e\}$.

⑦ $E + E^\infty = E^\infty$

$$E^\infty = \bigoplus_{n\in\mathbb{N}} E$$

$$\boxed{V \oplus V^\infty \cong V^\infty}$$

$$V^\infty = \bigoplus_{n\in\mathbb{N}} V$$

(8) $V \rightsquigarrow V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ $\quad \lambda \cdot (\mu \otimes v) := \lambda\mu \otimes v.$

$\lambda \in \mathbb{C}_{\mathbb{R}}$, $\mu \in \mathbb{C}_{\mathbb{R}}$, $v \in V$

$= V \oplus V$

$i \cdot (u \oplus v) := (-v \oplus u)$

$V \rightsquigarrow V_{\mathbb{R}}$

If $\{v_j\}$ is a basis for V. Then $\{v_j\} \cup \{i v_j\}$ is a basis for $V_{\mathbb{R}}$.

↳ and also for v.bs.

a) $\operatorname{rank} E_{\mathbb{C}} = \operatorname{rank} E$

$\operatorname{rank} F_{\mathbb{R}} = 2 \operatorname{rank} F.$

(9) b) What is $(E_{\mathbb{C}})_{\mathbb{R}} \cong E \oplus E$

$(F_{\mathbb{R}})_{\mathbb{C}} \cong F \oplus F$

$\{v_j\} \rightsquigarrow \{v_j\} \cup \{i v_j\}$

$\rightsquigarrow$

$$KO(X) = Gr\left(\mathrm{Vect}_{\mathbb{R}}(X) / \sim_s \right)$$

$\downarrow \mathbb{C}$

$$KO(X) \xrightarrow{\mathbb{C}} K(X)$$

$$K(X) \xrightarrow{\mathbb{R}} KO(X)$$

Are these ring homs?

what is the composite.

$$KO(X) \xrightarrow{\mathbb{C}} K(X) \xrightarrow{\mathbb{R}} KO(X) \ ?$$

(and likewise the other way.)