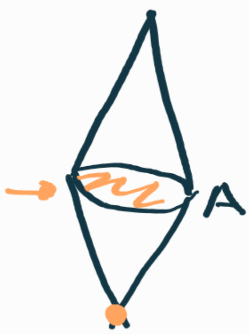
Deducing "Bott periodicity"

Let A be a closed subspace of a cpt. Hausdorff space X.

$$\begin{tikzcd}
A \arrow[r, hook] & X \arrow[r, hook] & X \cup CA \arrow[r, hook] \arrow[d, "\simeq"] & (X \cup CA) \cup CX \arrow[d] \arrow[r] & \cdots \\
 & & X/A & SA &
\end{tikzcd}$$

$$\begin{tikzcd}
\tilde{K}(A) & \tilde{K}(X) \arrow[l] & \tilde{K}(X \cup CA) \arrow[l] & \tilde{K}((X \cup CA) \cup CX) \arrow[l] & \cdots \arrow[l] \\
 & & \tilde{K}(X/A) \arrow[u] & \tilde{K}(SA) \arrow[u] &
\end{tikzcd}$$



Prop. If $A \xrightarrow{\iota} X \xrightarrow{q} X/A$ is an inclusion (then quotient) of a closed subspace of X (cpt. Haus.), then $\tilde{K}(A) \xleftarrow{\iota^*} \tilde{K}(X) \xleftarrow{q^*} \tilde{K}(X/A)$ is exact at $\tilde{K}(X)$.

Prop. In the special case that $A \subseteq X$ is contractible, then q^* is an isomorphism, and gives a well-defined iso

$$\mathrm{Vect}^n(X/A) \longrightarrow \mathrm{Vect}^n(X).$$

$$\Downarrow$$

$$[E]$$

Pf. Start with $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$. Well, E is at least trivial over $A \subseteq X$. Pick a trivialization h.

The recipie we have already seen lets us build

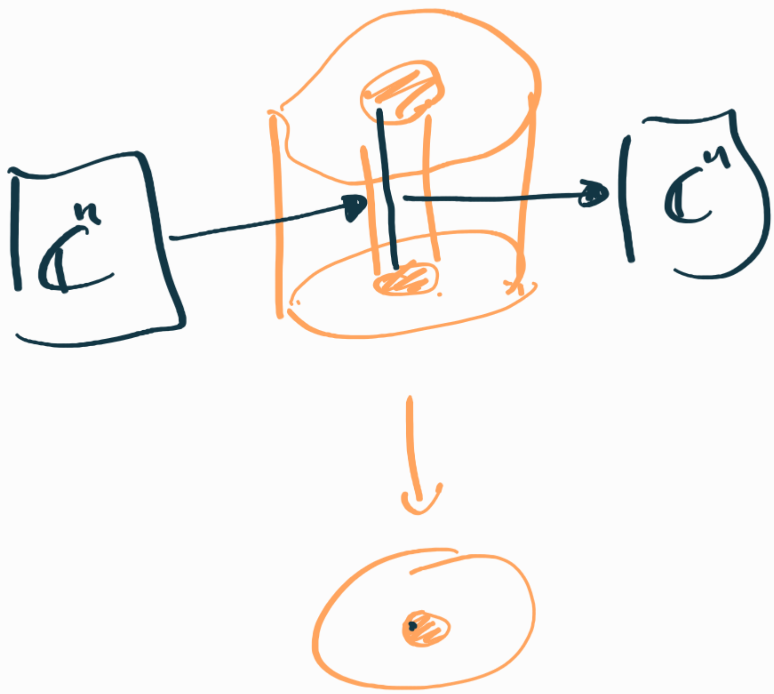
$$\begin{matrix} E/h \\ \downarrow \\ X/A \end{matrix} \quad \checkmark$$

Let's consider a second choice h' of loc. triv. of E over A. Then we can form the composite

$$f = h' \circ h^{-1} : A \longrightarrow GL(\mathbb{C}^n).$$

Observe that this lets us write

$$h' = (h' \circ h^{-1})\, h .$$



Now use that A is contractible to conclude that f is homotopic to a constant map valued at $L \in GL(\mathbb{C}^n)$.

Let $F : I \times A \longrightarrow GL(\mathbb{C}^n)$ be that homotopy.

Then $h_t = F_t h$ is a family of loc. triv.

with $h_0 = f h = h'$, and $h_1 = L h$.

It now follows from our general machinery that the bundles

$$E/_{h'} = E/_{h_0} \quad \text{and} \quad E/_{h_1} = E/_{Lh} \quad \text{are}$$

isomorphic. (Form $E \times I/_{h_t}$.)

Plus, $E/_{Lh} \cong E/_{h}$ via an obvious, explicit map.

Putting these together we conclude $[E/_{h}] = [E/_{h'}]$, and hence we have a well-defined map

$$\mathrm{Vect}^n(X) \longrightarrow \mathrm{Vect}^n(X/A).$$

Finally, we can draw a picture

$$\begin{tikzcd}
E/h \arrow[r, "\tilde{q}^*"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\
X/A \arrow[r, "q"] & X
\end{tikzcd}$$

Observe that $(*)$ is a fiberwise isomorphism.

So the pair $(p', \tilde{q})$ give an isomorphism between E/h and q^*E.

$\blacksquare$

Upshot, the sequence from before (of $\tilde{K}(-)$ groups) is a LES:

$$\begin{tikzcd}
 & & \tilde{K}(X \cup CA) \arrow[dl] \arrow[d, "\wr s"] & \tilde{K}((X \cup CA) \cup CX) \arrow[l] \arrow[d, "\wr s"] \\
\tilde{K}(A) & \tilde{K}(X) \arrow[l] & \tilde{K}(X/A) \arrow[l] & \tilde{K}(SA) \arrow[l] \arrow[ul] \arrow[ll, bend right]
\end{tikzcd}$$

$$\longleftarrow \tilde{K}(SX) \longleftarrow \tilde{K}(S(X/A)) \longleftarrow \cdots$$

Ex. Suppose $X = A \vee B$. (Noting that $X/A = B$), we get

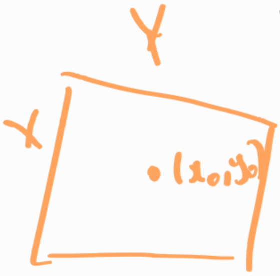
$$\tilde{K}(A) \longleftarrow \tilde{K}(X) \longleftarrow \tilde{K}(B) \xleftarrow{\;0\;} \tilde{K}(SA) \longleftarrow \cdots$$

$$\tilde{K}(X) \cong \tilde{K}(A) \oplus \tilde{K}(B).$$

<u>Upshot 2:</u> Our external product descends to the level of reduced K groups.

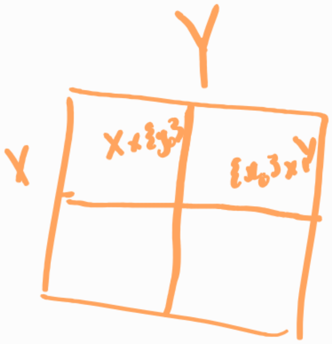
For <u>pointed</u> topological spaces, the:

- <u>product</u> $(X, x_0) \times (Y, y_0)$



$$= (X \times Y, (x_0, y_0))$$

- <u>tensor product</u> $(X, x_0) \otimes (Y, y_0)$



$$= (X \wedge Y, [(x_0, y_0)])$$

The smash product.

<u>Recall:</u> The ordinary external prod. of

$$\begin{array}{ccc} E & & F \\ \downarrow & \& & \downarrow \\ X & & Y \end{array}$$

was $pr_1^* E \otimes pr_2^* F$.

$$\begin{tikzcd} & X \times Y \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ X & & Y \end{tikzcd}$$

Thus we had

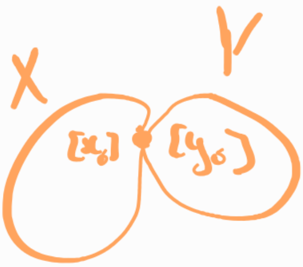
$$K(X) \otimes K(Y) \to K(X \times Y).$$

$$X \wedge Y = X \times Y \,/\, X \vee Y$$

the basepoint.

$$= \frac{X \times Y}{(X \times \{y_0\}) \cup (\{x_0\} \times Y)}$$

· <u>coproduct</u> $(X, x_0) \vee (Y, y_0)$



$$= (X \vee Y, [x_0] = [y_0]).$$

Really we should view our $\tilde{K}$-grps as functors defined on this cat.

$$\tilde{K}(X) := \ker\big(K(X) \longrightarrow K(x_0)\big)$$

We have $X \vee Y \longrightarrow X \times Y \longrightarrow X \wedge Y$, and so the LES

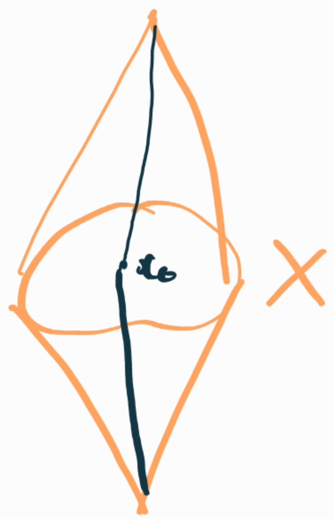
$$\begin{tikzcd}
\tilde{K}(X \vee Y) \arrow[d, phantom, "\cong"] & \tilde{K}(X \times Y) \arrow[l] & \tilde{K}(X \wedge Y) \arrow[l] & \tilde{K}(S(X \vee Y)) \arrow[l] \arrow[d, phantom, "\cong"] \\
\tilde{K}(X) \oplus \tilde{K}(Y) & \text{splitting.} & & \text{Claim: } \tilde{K}(SX) \oplus \tilde{K}(SY)
\end{tikzcd}$$

$$\leftarrow \tilde{K}(S(X\times Y)) \leftarrow \cdots$$

Pf. of Claim. Recall that we have a reduced suspension $\Sigma(X,x_0) = SX \big/ I\times\{x_0\}$.

We again have that

$\tilde{K}(\Sigma X) \xrightarrow{\sim} \tilde{K}(SX)$ is an iso, because $I\times\{x_0\}$ always contractible.



So, it suffices to check that

$$\tilde{K}(\Sigma(X\vee Y)) \cong \tilde{K}(\Sigma X)\oplus\tilde{K}(\Sigma Y).$$

or in turn

$$\Sigma(X\vee Y) = \Sigma X \times \Sigma Y.$$

Ex. Check this.

∎

pr_1^* gives a map $\tilde{K}(X) \to \tilde{K}(X \times Y)$, likewise

for pr_2^* ——————— $\tilde{K}(Y) \to \tilde{K}(X \times Y)$.

In fact these maps assemble into a splitting $pr_1^* \oplus pr_2^*$ of $\tilde{K}(X \times Y) \to \begin{array}{c} K(X) \\ \oplus \\ K(Y) \end{array}$.

↳ e.g. starting with $\begin{array}{c} E \\ \downarrow \\ X \end{array}$, pr_1^* lets us build $\begin{array}{c} pr_1^* E \\ \downarrow \\ X \times Y \end{array}$, and then we can restrict along $X \vee Y \overset{\iota}{\hookrightarrow} X \times Y$, obtaining $\begin{array}{c} \iota^* pr_1^* E \\ \downarrow \\ X \vee Y \end{array}$.

But $\tilde{K}(X \vee Y) \to \tilde{K}(X)$ is induced by $X \underset{\iota_X}{\hookrightarrow} X \vee Y$, so the resulting class in $\tilde{K}(X)$ is $\iota_0^* \iota^* pr_1^* E = E$ $\quad \iota_1^* \iota^* pr_2^* E = \varepsilon^0$.

That is, $pr_1^* \oplus pr_2^*$ splits i^*

$i : X \vee Y \hookrightarrow X \times Y$

It follows we have isos.

$$\tilde{K}(X \times Y) \cong \tilde{K}(X) \oplus \tilde{K}(Y) \oplus \tilde{K}(X \wedge Y).$$

$$\left(\tilde{K}(S(X \times Y)) \cong \tilde{K}(SX) \oplus \tilde{K}(SY) \oplus \tilde{K}(S(X \wedge Y)) \right)$$

Now, the external product fits into

$$\begin{tikzcd}
K(X) \otimes K(Y) \arrow[d, "\mu"'] & \cong & \left(\tilde{K}(X) \otimes \tilde{K}(Y)\right) \arrow[d, "\mu"'] & \oplus & \tilde{K}(X) \arrow[d, "id"] & \oplus & \tilde{K}(Y) \arrow[d, "id"] & \oplus & \mathbb{Z} \arrow[d, "id"] \\
K(X \times Y) & \cong & \tilde{K}(X \wedge Y) & \oplus & \tilde{K}(X) & \oplus & \tilde{K}(Y) & \oplus & \mathbb{Z}
\end{tikzcd}$$

We define $\mu : \tilde{K}(X) \otimes \tilde{K}(Y) \longrightarrow \tilde{K}(X \wedge Y)$ to be the induced map.

Upshot: we can take external products of elements of reduced K groups.

Thm. (Bott theorem). There is a canonical isomorphism

$$\tilde K(X) \longrightarrow \tilde K(S^2 X).$$
$$S^2 X = S(SX)$$

Pf. The iso is the composite.

$$\tilde K(X) \xrightarrow[\sim]{1\otimes -} \begin{array}{c} \mathbb{Z} \overset{\text{FPT}}{\cong} \tilde K(S^2) \\ \otimes \quad 1 \mapsto H-1 \quad \otimes \\ \tilde K(X) = \tilde K(X) \end{array} \xrightarrow[\sim \ \text{FPT}]{\mu} \tilde K(S^2 \wedge X) = \tilde K(\Sigma^2 X) \xrightarrow{s} \tilde K(S^2 X).$$

∎

Unfolding defn.s note that

$$S^1 \wedge X = \Sigma X.$$

In the direction of turning $\tilde{K}$ into a <u>cohomology theory</u>, inspired by the periodicity, let's define

$$\tilde{K}^{-n}(X) = \tilde{K}(S^n X).$$

& $$\tilde{K}^{-n}(X,A) = \tilde{K}(S^n(X/A)).$$

$$\begin{tikzcd}
\tilde{K}^0(A) & \tilde{K}^0(X) \arrow[l] & \tilde{K}^0(X,A) \arrow[l] & \tilde{K}^{-1}(A) \arrow[l] & \tilde{K}^{-1}(X) \arrow[l] & \tilde{K}^{-1}(X,A) \arrow[l] \\
 & & & & \tilde{K}^0(A) \arrow[r, "\sim"', "\cdot(H-1)" below] & \tilde{K}^{-2}(A) \arrow[u] \\
 & & & & \tilde{K}^0(X) \arrow[u] \arrow[r, "\sim", "\cdot(H-1)"'] & \tilde{K}^{-2}(X) \arrow[u] \\
 & & & & & \vdots \arrow[u]
\end{tikzcd}$$

We get a six-term exact sequence for complex K-theory:

$$\begin{tikzcd}
\tilde{K}^0(X) \arrow[r] & \tilde{K}^0(A) \arrow[r] & \tilde{K}^{-1}(X,A) \arrow[d] \\
K^0(X,A) \arrow[u] & \tilde{K}^{-1}(A) \arrow[l] & \tilde{K}^{-1}(X) \arrow[l]
\end{tikzcd}$$

(People also just define $\tilde{K}^n(X) = \begin{cases} \tilde{K}(X) & n \text{ even} \\ \tilde{K}(SX) & n \text{ odd} \end{cases}$ $n>0$

ptd. cpt. Hausdorff

<u>Prop.</u> Suppose X is the union of closed subspaces A and B. Then if A & B are each contractible, the product in $\tilde{K}^*(X)$ is zero.

e.g.

Pf. The map $\tilde{K}^*(X) \otimes \tilde{K}^*(X) \longrightarrow \tilde{K}^*(X)$ is just the composite

$$\tilde{K}^*(X,A) \otimes \tilde{K}^*(X,B) \longrightarrow \tilde{K}^*(X,A$$

<u>Pause.</u>

We can compute $\tilde{K}^{0,1}(\mathbb{RP}^2)$,

$\tilde{K}(\mathbb{RP}^4/\mathbb{RP}^2)$.

We at least know $\tilde{K}(S_1)$.

If we start with $\mathbb{RP}^1 \hookrightarrow \mathbb{RP}^2$, what does the 6-term exact sequence say?

$$\begin{tikzcd}
\tilde{K}^0(X) \arrow[r] & \tilde{K}^0(A) \arrow[r] & \tilde{K}^{-1}(X,A) \arrow[d] \\
K^0(X,A) \arrow[u] & \tilde{K}^{-1}(A) \arrow[l] & \tilde{K}^{-1}(X) \arrow[l]
\end{tikzcd}$$

$\mathbb{Z}$

$\| S$

$$\begin{tikzcd}
\tilde{K}^0(\mathbb{RP}^2) \arrow[r] & \tilde{K}^0(\mathbb{RP}^1) \arrow[r] & \tilde{K}^0(\mathbb{RP}^2/\mathbb{RP}^1) \arrow[d] \\
\tilde{K}^{-1}(\mathbb{RP}^2,\mathbb{RP}^1) \arrow[u] & \tilde{K}^{-1}(\mathbb{RP}^1) \arrow[l] & \tilde{K}^{-1}(\mathbb{RP}^2) \arrow[l]
\end{tikzcd}$$

$\tilde{K}(S(S^1)) = 0$

$\|$ $\tilde{K}^{-1}(\mathbb{RP}^1)$