Week 1 Problems

June 24, 2022

Problem 1. Which condition do we need on the transition functions $t_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^n)$ so that together they assemble into a vector bundle?

Problem 2. What condition do we need on maps $f_\alpha : U_\alpha \to \mathrm{GL}(\mathbb{R}^m \to \mathbb{R}^n)$ so that they assemble into a map of vector bundles?

Problem 3. Let $f : E \to F$ of be a map of vector bundles over the same base B. Prove that if f restricts to an isomorphism on each fiber then f is an isomorphism, via the following strategy:

1. Show that E and F have a common trivializing open cover $\{U_\alpha\}$.
2. Reduce to the case that E and F are both trivial by showing that it suffices to check that f is an isomorphism when restricted to a map of vector bundles $f : E_{U_\alpha} \to F_{U_\alpha}$.
3. Prove the claim in this special case.

Problem 4. Recall the Möbius bundle $M \to S^1$ from Day 1. Compute $M \oplus M$, $M \otimes M$, and (if we had time to define it) the pullback $\gamma_n^* M$ where $\gamma_n : S^1 \to S^1$ is the standard winding number n loop.

Problem 5. Show that our definition of the direct sum of vector bundles coincides with Hatcher's.

Problem 6. Show that TS^1 is trivial. *Hint: view S^1 as a subset of $\mathbb{C}$ and consider the action of multiplication by i. (Could we make something similar work for e.g. TS^3?)*

Problem 7. Show that TS^2 is *stably trivial*[1].

Problem 8. Show that a vector bundle $E \to B$ has k linearly independent sections if and only if E has a trivial k-dimensional subbundle.

Problem 9. Show that the orthogonal complement of a subbundle is independent (up to isomorphism) of the choice of inner product.

[1]Recall that this means that TS^2 becomes trivial after taking a direct sum with a trivial bundle.