Week 2 Problems

July 1, 2022

For additional context see the online notes for this class.

Just as we have used $\mathrm{Vect}^k(B)$ to denote the set of isomorphism classes of rank k real vector bundles over B, let $\mathrm{Vect}^k_{\mathbb{C}}(B)$ denote the isomorphism classes of rank k complex vector bundles over B.

Problem 1. Check that the conjugation[1] operation on complex vector spaces gives rise to a conjugation operation for complex vector bundles, which induces an automorphism of the set $\mathrm{Vect}^k_{\mathbb{C}}(B)$.

Problem 2. Recall the definition of an inner product on a real vector bundle $E \to B$. State the analogous definition of a (Hermitian) inner product on a complex vector bundle.

Problem 3. Recall that if $L \to B$ is a real line bundle then $L \otimes L$ is always trivial. Using a similar argument, show that if L is instead a complex line bundle then $L \otimes \overline{L}$ is always trivial.

Let's write $\varepsilon_n \to B$ for the rank n trivial complex vector bundle over B.

Problem 4. Show that TS^2 is *stably trivial*, in particular by exhibiting an isomorphism between $TS^2 \oplus \varepsilon_1$ and ε_3.

Let $\sim_{\mathrm{s}}$ denote the relation on $\mathrm{Vect}^k_{\mathbb{C}}(B)$ (or $\mathrm{Vect}^k(B)$) of *stable isomorphism*, i.e. $[E] \sim_{\mathrm{s}} [F]$ iff there is some $n > 0$ such that $E \oplus \varepsilon_n \cong F \oplus \varepsilon_n$. Similarly declare that $[E] \sim [F]$ if there are $m, n > 0$ such that $E \oplus \varepsilon_m \cong F \oplus \varepsilon_n$.

Problem 5. Check that $\sim_{\mathrm{s}}$ and $\sim$ are well-defined equivalence relations.

Problem 6. Last time we computed $M \oplus M$ and $M \otimes M$ for $M \to S^1$ the Möbius bundle; now compute the pullback $\gamma_n^* M$ where $\gamma_n : S^1 \to S^1$ is the standard winding number n loop.

Definition 7. When B is compact Hausdorff, the quotient $\tilde{\mathrm{K}}(B) := \mathrm{Vect}^k_{\mathbb{C}}(B)/\!\sim$ is the *reduced K-group* of B.

Problem 8. Verify that $\tilde{\mathrm{K}}(B)$ is an abelian group.

Definition 9. Let $(M, +)$ be a commutative monoid with identity $0 \in M$. The *group completion* $\mathrm{Gr}(M)$ of M is the quotient of $M \times M$ by the relation which identifies (m_1, m_2) with (m_1', m_2') exactly when $m_1 + m_2' = m_1' + m_2$.

One is supposed to think of the class $[(m_1, m_2)] \in \mathrm{Gr}(M)$ as the formal difference "$m_1 - m_2$".

[1] See the preliminary notes for a definition.

Problem 10. Check that $\mathrm{Gr}(M)$ is canonically a group under the operation $[(m_1, m_2)] + [(m_1', m_2')] = [(m_1 + m_1', m_2 + m_2')]$ and identity $[(0,0)]$. Show that the function $M \to \mathrm{Gr}(M)$ defined by $m \mapsto [(m, 0)]$ is a monoid homomorphism.

Definition 11. When B is compact Hausdorff, the group completion $\mathrm{K}(B) := \mathrm{Gr}(\mathrm{Vect}^k_{\mathbb{C}}(B)/{\sim_s})$ is the *K-group* of B.

Problem 12. Show that there is a surjective group homomorphism $\mathrm{K}(B) \to \widetilde{\mathrm{K}}(B)$ defined by $[(E, F)] \mapsto [E] - [F]$. What is the kernel of this map?

Definition 13. For each continuous map $f : B' \to B$ let $\mathrm{K}(f)$ be the function $\mathrm{K}(B') \to \mathrm{K}(B)$ induced by the pullback by f, i.e. defined by $[(E, F)] \mapsto [(f^*E, f^*F)]$.

Problem 14. Show that this definition of $\mathrm{K}(f)$ gives a well-defined group homomorphism for each continuous function f. Moreover, verify that $\mathrm{K}(f)$ depends only on the homotopy class of f.

Problem 15. Show that $\mathrm{K}(B)$ is a ring under the multiplication operation induced by the tensor product of vector bundles, and that with respect to this operation $\mathrm{K}(f)$ is a ring homomorphism.

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Problem 16. Show that a vector bundle $E \to B$ has k linearly independent sections if and only if E has a trivial k-dimensional subbundle.

Problem 17. Show that the orthogonal complement of a subbundle is independent (up to isomorphism) of the choice of inner product.