Week 3 Problems

July 8, 2022

Problem 1. Let $M \to S^1$ be the Möbius bundle, and let $\gamma_n : S^1 \to S^1$ be the standard winding number n loop. Show that the pullback $\gamma_n^* M$ is trivial for n even and isomorphic to M for n odd.

Problem 2. Show that $\mathrm{GL}(\mathbb{C}^n)$ is connected. Why does the same argument fail for $\mathrm{GL}(\mathbb{R}^n)$?

Problem 3. Recall that $\mathbb{C}\mathrm{P}^n$ is the space of lines in $\mathbb{C}^{n+1}$. Convince yourself that $\mathbb{C}\mathrm{P}^1$ is homeomorphic to S^2.

Problem 4. Convince yourself of the argument of Example 1.10 of Hatcher (which we essentially saw last time), that the canonical line bundle $H \to \mathbb{C}\mathrm{P}^1$ has clutching function $f(z) = z$.

Problem 5. Show that $(H \otimes H) \oplus 1 = H \oplus H$ in $K(S^2)$ by computing clutching functions for the left and right hand sides.

Problem 6. Define $\mathbb{N}_\infty = \{0, 1, 2, \ldots, \infty\}$. Give $\mathbb{N}_\infty$ the structure of a commutative monoid by extending the operation of addition on $\mathbb{N}$ by declaring that $n + \infty = \infty = \infty + n$ for all $n \in \mathbb{N}_\infty$. Compute the group completion $\mathrm{Gr}(\mathbb{N}_\infty)$.

Problem 7. Let $\mathbf{Vect}(X)$ be the set of all isomorphism classes of *possibly infinite dimensional*[1] (complex) vector bundles over X. Once again calculate the group completion $\mathrm{Gr}(\mathbf{Vect}(X))$.

Problem 8. Show that the complexification $V_\mathbb{C}$ of ordinary real vector spaces V gives rise to a corresponding operation which turns real vector bundles $E \to B$ into complex ones $E_\mathbb{C} \to B$. What is the rank of $E_\mathbb{C}$ in terms of the rank of E?

Likewise show that realification $V_\mathbb{R}$ of complex vector spaces V again gives rise to a corresponding operation on complex vector bundles.

Denote the real K-group of a space X by $\mathrm{KO}(X)$.

Problem 9. Show that realification and complexification induce maps $\mathrm{Vect}_\mathbb{R}(X) \rightleftarrows \mathrm{Vect}_\mathbb{C}(X)$ which descend to group homomorphisms $\mathrm{KO}(X) \rightleftarrows \mathrm{K}(X)$ of K-groups. Of course, $\mathrm{K}(X)$ and $\mathrm{KO}(X)$ are also both rings; are these maps ring homomorphisms?

Finally, the two possible composites of these maps give endomorphisms of the groups $\mathrm{KO}(X)$ and $\mathrm{K}(X)$; compute them.

Problem 10. Show that every real vector bundle over a compact base is a subbundle of a trivial bundle.

[1] Q: Formally, what should this mean?