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1 Introduction

My work focuses on symplectic geometry and mirror symmetry, and I am interested broadly in capturing geometric and categorical phenomena detected in mathematical physics. In particular, I am currently investigating Fukaya categories and the family Floer approach to Kontsevich's homological mirror symmetry conjecture. The Fukaya category of a symplectic manifold is an algebraic package—an example of an A_∞-category—which encapsulates its Lagrangian Floer theory. My thesis develops a new model of family Floer theory via Morse theoretic technology, for the purpose of proving Theorem 1.1.

Theorem 1.1. *There is a functor of curved A_∞-categories*

$$C : \mathcal{F}_{\text{sec}} \to \text{mod-}\mathcal{A}$$

from the Fukaya category of Lagrangian sections of a (suitable) SYZ fibration $\pi : X \to B$ into the category of modules over the Morse–Fukaya algebra $\mathcal{A}$ of the fibration π.

The Strominger–Yau–Zaslow (SYZ) conjecture [13], and homological mirror symmetry (HMS) as originally set out by Kontsevich [10], are conjectures spawned from the mirror symmetry phenomenon observed by string theorists. Roughly speaking and as originally understood, SYZ mirror symmetry begins with a fibration of some kind of Kähler manifold by Lagrangian tori, and builds from this information a dual torus fibration via a geometric recipe. The difficulty is that the original fibration is allowed to have singular fibers (and generally will), and so the constructed mirror must be deformed accordingly.

On the other hand, homological mirror symmetry asserts a derived equivalence between the Fukaya category of a symplectic manifold and the category of coherent sheaves on its mirror [11]. Though in general it must be decided—as HMS is extended much beyond its initial incarnation comparing Calabi–Yau manifolds with their honest Calabi–Yau mirrors—what precisely is meant by Fukaya category, and whether to replace the derived category of coherent sheaves with for instance a noncommutative analogue.

The family Floer program [7, 1, 2, 14, 16] gives a modern reinterpretation of the construction of the SYZ mirror of $\pi : X \to B$ as a moduli space of objects of the Fukaya category of X supported on the fibers of the fibration. The resulting object, technically a rigid analytic mirror $X^\vee$ of X, comes equipped with a functor from the Fukaya category of X into coherent sheaves on $X^\vee$ which can then, as an application, be used to prove HMS as asserted.

The Morse–Fukaya algebra $\mathcal{A}$ of the fibration $\pi : X \to B$ is an A_∞-algebra determined by a Morse function on the total space X, taking coefficients in analytic functions on its mirror. For an appropriate choice of Morse function, $\mathcal{A}$ can be understood as a (suitably deformed) algebra of Čech cochains valued in polyvector fields on the SYZ mirror of X. The functor I construct in Theorem 1.1 then gives an analogous presentation of this story; here the category mod-$\mathcal{A}$ plays the role of a category of coherent sheaves as we explain below.

2 The Morse–Fukaya algebra of an SYZ fibration

Throughout fix a Kähler manifold (X, ω, J) with symplectic form ω and almost complex structure J. Let $\pi : X \setminus D \to B$ be a fibration of X by Lagrangian tori, where $D \subset X$ is a complex hypersurface representing the anticanonical class of X. The standard way to proceed is to first construct a mirror to the complement $X^0 = X \setminus D$.

As an act of technical expediency let us fix any compact, simply connected subset $B^0 \subset B$ which is disjoint from the critical values of π, and let $X^{00} = \pi^{-1}(B^0) \subset X^0$ be the corresponding restriction of the total space. For now also assume that $f : X^{00} \to \mathbb{R}$ is any Morse–Smale function with respect to a choice of metric on X^{00}.

The basic objects we consider are *(pseudoholomorphic) treed disks*; geometrically, these are continuous maps $u : \Delta \to X^{00}$ of decorated domains Δ built from the complex unit disk via an inductive gluing procedure, and which satisfy relations determined by their decorations (as developed by Charest–Woodward in [4] and originally Cornea–Lalonde [5]). Namely, an additional copy C of the unit disk may be glued into Δ by attaching one endpoint of a new line segment to the boundary ∂C, and the other endpoint to the boundary of a disk already in Δ. We call the image of each such disk C in Δ a *disk component*. It is also desirable to permit the attachment of semi-infinite line segments (rays) to disk boundaries, and to always remember the orientation of line segments we attach (whether finite or semi-infinite). Figure 1 depicts a schematic diagram of a treed disk domain built from two complex unit disks and four line segments (three of the segments having open ends).

Write $S \subset \Delta$ for the interior of the disks in Δ (the *surface part*) and $T \subset \Delta$ for the interior of the attached line segments (the *tree part*); then u restricts to maps $u_S(x)$ and $u_T(t)$ defined on S and T respectively. Let j be the complex structure on S induced by the standard complex structure on the unit disk.

Definition 2.1. A continuous map $u : \Delta \to X^{00}$ with treed disk domain Δ is *pseudoholomorphic* if we have both

(2.I) u is *pseudoholomorphic* on the surface part: $J \circ \mathrm{D}\, u_S = \mathrm{D}\, u_S \circ j$, and

(2.II) u is a *Morse gradient flow line* on the tree part: $\dfrac{\mathrm{d}u_T}{\mathrm{d}t} = \nabla f \circ u_T$.

In other words, u must consist of a family of pseudoholomorphic disks attached along their boundaries, according to the edges of a tree, via Morse gradient flow lines.

Note that we have already suppressed several technical details; for example, in practice we perturb the pseudoholomorphic curve equation (2.I) due to transversality issues which arise while setting up the theory. In general, we allow J to be a domain-dependent almost complex structure determined by a background system of perturbation data, and similarly for (2.II). This data is chosen and managed consistently via an extension of the scheme of Charest–Woodward [4] (using stabilizing divisors) to the family setting. Relatedly, it is often convenient to equip points of treed disk domains with certain combinatorial bookkeeping labels, but we suppress these here as well.

Pseudoholomorphic treed disks give rise to algebraic operations via fixing a family of domains, prescribing boundary conditions, and then taking signed counts of their zero

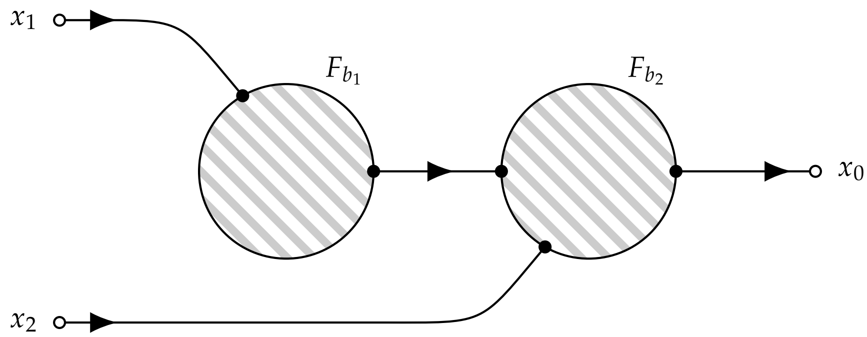


Figure 1: A schematic diagram of a pseudoholomorphic treed disk.

dimensional (compact) moduli spaces. These algebraic operations act on a complex with Morse theoretic generators, and this complex takes coefficients in a sheaf $\mathcal{O}_{\text{an}}$ obtained from the analytic functions on the *uncorrected mirror* of X^0. Let us now describe each of these details.

First, it is not difficult for us to arrange that the Morse function f on X^{00} lifts a Morse function on B^0 (so that their critical points coincide under π), and that f restricted to each fiber is perfect (i.e. gives a minimal Morse model for the n-torus). Let P be a cellular decomposition of B^0 with the property that each k-cell $\sigma \in P^{[k]}$ contains in its interior a unique index k critical point b_σ, and that σ is itself the closure of the stable manifold of b_σ.

Writing $F_b = \pi^{-1}(b)$ for the fibers, the uncorrected mirror then has underlying set of points simply the disjoint union [11, 1]

$$X^{\vee 0} = \bigsqcup_{b \in B_0} \mathrm{H}^1(F_b, U_\Lambda)$$

with $U_\Lambda = \mathrm{val}^{-1}(0)$ the unitary subgroup of the Novikov field Λ we work over.[1] Identify the groups $\pi_2(X, F_b)$ via isotoping fibers; then for each $\beta \in \pi_2(X, F_b)$ we naturally obtain a function

$$z^\beta = T^{\omega(\beta)} \,\mathrm{hol}(\partial\beta)$$

on $X^{\vee 0}$. In the definition of this *Floer-theoretic weight*, $\omega(\beta)$ is the symplectic area of β and $\mathrm{hol}(\partial\beta)$ denotes[2] evaluation of points of $X^{\vee 0}$ on the class $\partial\beta$. The set $X^{\vee 0}$ is naturally endowed with the structure of a rigid analytic space (having $(\Lambda^*)^n$ as a local model) for

[1]The Novikov field with coefficients in the field $\Bbbk$ (of characteristic zero, which we fix throughout) consists of series in the formal variable T of the form

$$\Lambda = \left\{ \textstyle\sum_i c_i T^{x_i} : c_i \in \Bbbk, x^i \in \mathbb{R}, x_i \to \infty \right\},$$

and comes equipped with a valuation map $\mathrm{val} : \sum_i c_i T^{x_i} \mapsto \min\{x_i : c_i \neq 0\}$.

[2]This notation is due to the fact that $X^{\vee 0}$ is realized as a moduli space of fibers of π equipped with a unitary rank 1 local system—since points of which are determined by their holonomy map, they equivalently belong to $\mathrm{H}^1(F_b, U_\Lambda)$ for some $b \in B$.

which the functions z^β are analytic. A chart is furnished by $(z^{\beta_i})_{1\le i\le n}$, with the classes β_i chosen so that the $\partial\beta_i$ form a basis of $\mathrm{H}_1(F_b;\mathbb{Z})$.

There is a natural projection $\pi^\vee : X^{\vee 0} \to B^0$ and, after suitably refining P by perturbing f, the collection of analytic functions on $\pi^{-1}(\mathrm{star}(\sigma))$ for each $\sigma \in P$ assemble into a sheaf $\mathcal{O}_{\mathrm{an}} = \pi^\vee_*(\mathcal{O}_{X^{\vee 0}})$ of universal weights.

Definition 2.2. The *Morse–Fukaya algebra* $\mathcal{A} = \mathrm{CM}^\bullet(\pi, f; \mathcal{O}_{\mathrm{an}})$ is the module freely generated by the critical points of f, with coefficients taken in $\mathcal{O}_{\mathrm{an}}$, equipped with algebraic operations μ^d for $d \ge 0$ defined below.

Fix $d+1$ points $x_0, x_1, \dots, x_d \in \mathrm{crit}\, f$. We say that a pseudoholomorphic treed disk $u : \Delta \to X^{00}$ has *d inputs and* 1 *output* if, respecting orientations, in constructing Δ we attached exactly d copies of the ray $(-\infty, 0]$ and 1 copy of the ray $[0,\infty)$. Call $x \in X^{00}$ the input (resp. output) of the ray $R = (-\infty, 0] \subset \Delta$ (resp. $R = [0,\infty) \subset \Delta$) whenever $\lim_{|t|\to\infty} u|_R(t) = x$. Thus Figure 1 depicts a pseudoholomorphic treed disk with 2 inputs x_1 and x_2 and 1 output x_0. Note that a treed disk domain Δ with 1 output has a canonical ordering on its inputs induced by the orientation of the disk components of Δ.

For each $\beta \in \pi_2(X, F_{b_0})$ we may form the moduli space $\overline{\mathcal{M}}_{d+1}(x_0, \dots, x_d; \beta)$ from all (suitably perturbed) pseudoholomorphic treed disks $u : \Delta \to X^{00}$ with:

- Disk boundaries lying on fibers—each disk component $C \subset \Delta$ satisfies $u(\partial C) \subset F_b$ for some $b \in B$ (all possibly different).
- Representing class β—each disk component $C \subset \Delta$ gives rise to a class $[u|_C] \in \pi_2(X, F_b)$ hence in $\pi_2(X, F_{b_0})$, and we demand that the sum of all such classes is β.
- Correct I/O—we require that u has d inputs $x_1, \dots, x_d$ and 1 output x_0.
- Stable components—the map u obeys a family of straightforward technical conditions[3] which ensure we obtain a compact Hausdorff moduli space with the correct dimension.

We must of course also take care to develop a consistent scheme to orient these moduli spaces, though we do not elaborate further here on these technical details [15, 8].

Now given $\mathbf{x} = (x_1, \dots, x_d) \in \mathrm{crit}\, f$ we set

$$\mu^d(\mathbf{x}) := \sum_{\beta, x_0} \#\overline{\mathcal{M}}_{d+1}(x_0, x_1, \dots, x_d; \beta) \cdot z^\beta x_0, \tag{2.III}$$

where # is the signed count of oriented points, and the sum is taken over classes β and critical points x_0 for which the expected dimension of $\overline{\mathcal{M}}_{d+1}(x_0, x_1, \dots, x_d; \beta)$ is zero. In accordance with the $\mathbb{Z}_2$-grading induced by Morse index mod 2, upon declaring that each μ^d is $\mathcal{O}_{\mathrm{an}}$-linear we obtain a family of graded multiplication maps $\mu^d : \mathcal{A}^{\otimes d} \to \mathcal{A}[2-d]$.

[3] For instance, we require that each disk component on which u is constant must meet at least 3 line segments.

Theorem 2.3. *The operations μ^d endow the Morse–Fukaya algebra $\mathcal{A}$ with the structure of a curved A_∞-algebra [3]. In other words, for each $d > 0$ and homogeneous elements $a_1, \ldots, a_d \in \mathcal{A}$ of respective degrees $|a_i|$ we have the identity [12]*

$$0 = \sum_{m+n \le d} (-1)^{\heartsuit} \mu^{d-n+1}(a_1, \ldots, a_m, \mu^n(a_{m+1}, \ldots, a_{m+n}), a_{m+n+1}, \ldots, a_d), \tag{2.IV}$$

with $\heartsuit = (-1)^{m+\sum_{i=1}^{m}|a_i|}$.

One obtains a proof of Theorem 2.3 by a careful analysis of the boundary components of the higher dimensional strata of the moduli spaces $\overline{\mathcal{M}}_{d+1}(x_0, \ldots, x_d; \beta)$ we have just introduced; ultimately, the signed count of points on the boundary of a 1-dimensional oriented compact moduli space is zero. For example, when a Morse gradient flow line in a treed disk "breaks" (on the boundary of a moduli space) through an intermediate critical point, the treed disk naturally decomposes as the composition of two less complex treed disks, one stacked upon the other. All such possible decompositions appear as terms in (2.IV).

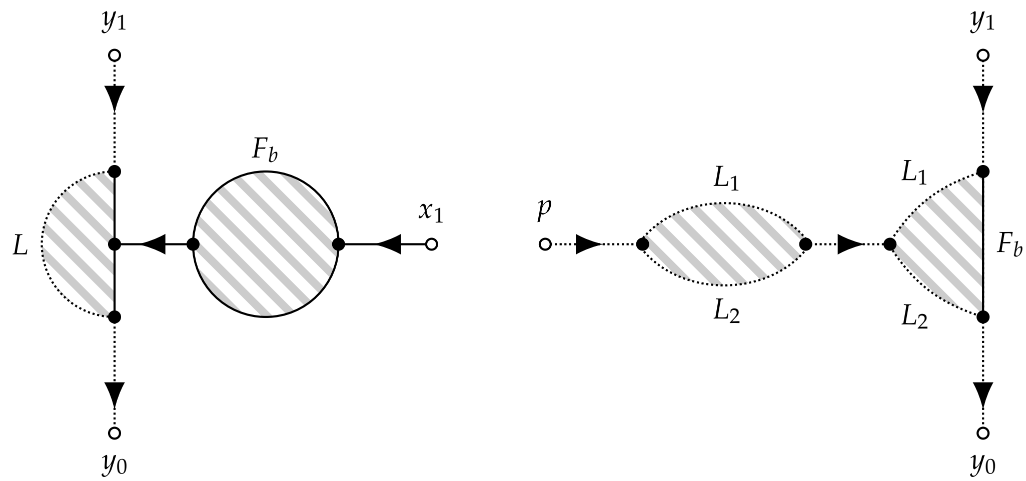
3 An HMS comparison functor

We associate to the Lagrangian fibration $\pi : X^{00} \to B^0$ a full subcategory $\mathcal{F}_{\text{sec}}$ of the Fukaya category of X, whose objects are Lagrangian sections L of π over B^0. Each such section is naturally equipped with a Morse function f_L via restriction of the global Morse function on X^{00}. For simplicity, fix a finite collection $\{L_i\} \subset \mathcal{F}_{\text{sec}}$ intersecting pairwise transversely. If $L_i \neq L_j$ we let $\text{Hom}(L_i, L_j)$ be freely generated by the points of $L_i \cap L_j$ with coefficients in Λ. If instead $L_i = L_j$ we substitute the Fukaya–Morse algebra of the ordinary Lagrangian $L = L_i$ as defined by Charest–Woodward [4] (i.e. $\text{Hom}(L, L)$ is generated by critical points of f_L with coefficients in Λ—the algebra operations are as above, except that we now require all Morse flow lines and disk component boundaries to lie wholly in L). The composition of, for example, morphisms $p \in \text{Hom}(L_1, L_2)$ and $q \in \text{Hom}(L_2, L_3)$ between distinct Lagrangian sections is the familiar multiplication in the Fukaya category; we count pseudoholomorphic strips with boundary on $L_1 \cup L_2 \cup L_3$ meeting p, q, and all possible third points of $\text{Hom}(L_1, L_3)$, in the usual way.

We are now in a position to see how the comparison functor $C : \mathcal{F}_{\text{sec}} \to \text{mod-}\mathcal{A}$ of Theorem 1.1 is defined. First, on objects we set

$$L \in \mathcal{F}_{\text{sec}} \mapsto C(L) := \text{CM}^\bullet(\pi, f|_L; \mathcal{O}_{\text{an}}),$$

this module being generated by the points of crit f_L with coefficients in $\mathcal{O}_{\text{an}}$. The object $C(L)$ carries a family of A_∞-module action maps $\triangleleft^d : C(L) \otimes \mathcal{A}^{\otimes d} \to C(L)$ which, according to the natural analogue of (2.III), now count moduli spaces of treed disks of the kind for example schematically depicted in Figure 2a. The key modification is that we now allow the boundary of disk components to lie on the union of a particular fiber and some number of Lagrangian sections; Morse gradient flow lines are in turn suitably constrained to either a fiber or particular sections.



(a) Coefficient of y_0 in $y_1 \triangleleft^1 x_1$ (b) Coefficient of y_0 in $C^1(p)(y_1)$

Figure 2: A pair of schematic diagrams of treed disks captioned with the coefficient to which their counts contribute in the associated module action or morphism. Boundaries and line segments constrained to a Lagrangian section are shown dotted.

Similarly, a morphism $p \in L_1 \cap L_2 = \mathrm{Hom}(L_1, L_2)$ gives rise to a morphism $C(L_1) \to C(L_2)$ of A_∞-modules by counting treed disks such as those schematically of the type depicted in Figure 2b or, for the higher order terms of the A_∞-module homomorphism, analogous configurations with additional inputs. Note that in this particular example both of the horizontal line segments in Figure 2b are designated Morse flow lines wholly contained in the transverse intersection $L_1 \cap L_2$—hence their image in any pseudoholomorphic treed disk must be constant. The requisite A_∞-relations for both the module actions and module morphisms hold by essentially the same analysis as in the previous section; we again carefully consider the several ways treed disks such as those schematically depicted in Figure 2 can break.

4 Future work

Direct natural continuations of this work are twofold; in each case a new insight is required to upgrade our technology. The first is to develop a method to treat general Lagrangians L in X which are not sections of the SYZ fibration. Without modification, ordinary counts of pseudoholomorphic strips with boundary on $L \cup F_b$ do not behave correctly when passing through a fiber F_b which has nontransverse intersections with L.

The second is to handle singular fibers, which currently must be excluded because of issues of convergence arising locally around the critical values. Namely, the natural completion used to obtain our sheaf of universal coefficients $\mathcal{O}_{\mathrm{an}}$ from local pieces fails to yield a ring of actual functions in a neighborhood of a singular fiber, and so must be suitably extended.

More broadly, we seek a suitable notion of noncommutative space as a means to conveniently globalize our constructions (and to package a mirror), formulated in a way which emphasizes geometric features. Also, under appropriate hypotheses and because there is an identification $\mathrm{H}^k(F_b, \mathbb{R}) \cong \bigwedge^k \mathrm{T}_b B$, the *Floer-theoretic obstruction* μ^0 of $\mathcal{A}$ may be viewed as corresponding to a Čech cochain $\mathbb{W} \in \mathrm{C}^\bullet(X^{\vee 0}; \bigwedge^\bullet \mathrm{T}_{X^{\vee 0}})$ valued in polyvector fields on $X^{\vee 0}$. It is expected (due to results of Fukaya [6] and Irie [9]) that both μ^0 and $\mathbb{W}$ each correspondingly satisfy an algebraic identity known as the *master equation*. I am investigating a promising approach to establish this relation in generality via a modification of the model described here which incorporates Auroux's [3] "spliced treed disks".

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