Monoidal ∞-categories over ∞-operads

Elements of *Higher Algebra*

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In this essay we rapidly motivate and introduce the theory of symmetric monoidal ∞-categories, bootstrapping off an intuitive reformulation of the case of symmetric monoidal 1-categories. We explain how to generalise our construction to a theory of monoidal ∞-categories over ∞-operads, and explicitly construct the special cases of the associative and commutative ∞-operads (in each case beginning with the analogous 1-categorical construction). The key definition of an algebra object of a generalised monoidal ∞-category over an ∞-operad is introduced and various notions of associative monoids are compared.

The family of little k-cube ∞-operads are constructed from their topological 1-categorical counterparts, in order to provide an infinite family of important examples into which the associative and commutative ∞-operads fit as special cases. We conclude by specialising to the particular stable symmetric monoidal ∞-category of spectra, explaining the payoff of the application of the general theory to this special case in stable homotopy theory—where in particular a robust and rich theory generalising commutative and homological algebra is made available for use as fundamental tools.

1 Generalising (non)commutative algebra; an example of an $\mathbb{E}_\infty$ ring

One great impetus for the development of symmetric monoidal ∞-categories comes from algebraic topology, and in particular, the study of cohomology theories which do not look like singular cohomology. This is for example a motivation of Jacob Lurie's program[1] of developing a theory of higher algebra through quasicategories, a variant of which we present here.

Consider perhaps the simplest common cohomology theory which is not singular cohomology; complex topological K-theory. Let Top be the category of sufficiently nice topological spaces, and let $\mathscr{H}$ denote the homotopy category of Top. It is an elementary fact that, at least for compact base spaces X, direct sum and tensor product of vector bundles over X confers the set of isomorphism classes of vector bundles over X a pair of compatible monoid structures. Passing to the Grothendieck completion we obtain a zeroth (extraordinary) cohomology group $K_0(X)$, which is also a *commutative ring*. We begin by seeking a natural generalisation; to equip the K_0 and higher functors themselves with a commutative ring-like structure, for which the fact that each $K_0(X)$ is a commutative ring will follows as a consequence.

[1] Of which substantial pieces are presented in [10].

By Brown's representability theorem [3] the reduced complex topological K-theory functors $\tilde{K}_n : \mathsf{Top} \to \mathsf{CommRing}$ are together each representable functors (when regarded as having domain category $\mathscr{H}$) satisfying

$$\tilde{K}_n(X) \cong \mathscr{H}(X \to E_n)$$

for some spectrum E called the *K-theory spectrum*. For example, one can concretely identify this E_0 for complex K-theory by first observing[2] that every n-dimensional complex vector bundle over some $X \in \mathsf{Top}$ is a pullback of some canonical bundle C_n over the *Grassman manifold* G_n. Moreover one can show that the map

$$\Phi_n : [X, G_n] \to \mathrm{Vect}^n_{\mathbf{C}}(X), \quad [f] \mapsto f^*(C_n)$$

is a natural bijection. These maps Φ_n for each dimension n can be assembled by taking the direct limit of the $G_n = BU(n)$,

$$\varinjlim_n BU(n) = BU$$

and we obtain a natural bijection (which almost descends to the Grothendieck completion of $K_0(X)$, but one must add a factor of $\mathbf{Z}$ in order to obtain the unreduced K-theory)

$$\tilde{K}_0(X) \cong [X, BU].$$

One additionally finds $K_1(X) \cong [X, U]$, and this completely determines the complex K-theory spectrum E by Bott periodicity.

The general consequences of the representability of K_0 (for example) are very interesting. For one thing, by the Yoneda lemma the classifying space BU is identified up to homotopy equivalence, but moreover the fact that K_0 maps into the category of commutative rings immediately equips $BU \in \mathscr{H}$ with the necessary categorical data specifying a *commutative ring object* of $\mathscr{H}$. That is, setting $R = BU$ there are functors

$$\alpha, m : R \times R \to R$$

encoding addition and multiplication, along with functors (with 1 the category with one object and one morphism)

$$\mathbb{0}, \mathbb{1} : 1 \to R$$

giving the additive and multiplicative identity, in addition to

$$- : R \to R$$

specifying an additive inverse. Requiring an object $R \times R$ in the definition, the commutative ring objects of $\mathscr{H}$ must be specified with respect to a symmetric monoidal structure on $\mathscr{H}$ (and in this case we take the obvious Cartesian product structure). All of this data together must satisfy the axioms of a commutative ring lifted to the level of categories (i.e. equalities become commuting diagrams—but commuting in $\mathscr{H}$ and therefore only holding up to homotopy).

It is an unfortunate fact that this categorical structure is not sufficient in order to formulate a theory of commutative algebra applicable to the setting. Practical attempts to do so lead to dead-ends, since the category $\mathscr{H}$ simply lacks[3] sufficient structure.

[2] A modern account beginning from first-principles is given in [5].
[3] A brief account is given in [10].

Dismayed, we might ask that BU be given the maximal amount of commutative ring structure—we could demand that the coherence homotopies are actually all identities, so that the topological space BU is really just a commutative ring on-the-nose. Obviously the standard theory of commutative algebra would apply to such an object! Once again unfortunately the answer is no, this time for the reason that we have demanded that $BU \in \mathscr{H}$ have *too much* structure. Indeed, the groups of a cohomology theory represented by a topological ring R can be directly calculated, and for instance the zeroth group is given by

$$h^0(X) = \prod_{n=0}^{\infty} H^n(X; \pi_n(R)),$$

which in particular cannot yield complex K-theory, for example.

In summary, the situation is that in K-theory we have algebraic laws coming from facts such as that tensor product of complex vector bundles distributes over direct sum (at least over compact base spaces), i.e. $E \otimes (F \oplus F') \cong (E \otimes F) \oplus (E \otimes F')$, but this only holds up to isomorphism and not outright. Nonetheless such isomorphisms are canonical in a stronger sense than that reflected in a commutative ring object structure in the symmetric monoidal category $\mathscr{H}$.

Our goal is thus to construct a theory in which this “up-to-coherent isomorphism” structure can safely reside; the theory of symmetric monoidal ∞-categories is one such framework. In this new language BU becomes an $\mathbb{E}_\infty$*-ring*, which in stable homotopy theory is also called a commutative ring spectrum.

The notion of an $\mathbb{E}_\infty$-ring transcends ∞-category theory, but in this world the $\mathbb{E}_\infty$-rings are the commutative algebra objects of a particular symmetric monoidal stable ∞-category. This circumvents the problems arising when applying the same notion in the case of a symmetric monoidal 1-category (which we have just seen). The ambient stable ∞-category happens to be the category Sp of structured ring spectra.

Therefore, we press onward with but a single objective—to answer the question:

What is an $\mathbb{E}_\infty$-ring?

2 Arriving at the definition; first, symmetric monoidal 1-categories

Our objective will be to rapidly motivate and then generalise the notion of a symmetric monoidal ∞-category, and we begin by briefly reviewing the structure of a monoidal 1-category. Fix such a category $\mathscr{C}$. Then $\mathscr{C}$ is a 1-category equipped with the data of

- a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ called the *monoidal product*,
- an object $\mathbb{1} \in \mathscr{C}$ called the *tensor unit*, and
- three natural isomorphisms, in components given by
  - $\alpha : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ and called the *associator*, and
  - $\lambda : \mathbb{1} \otimes X \to X$ and $\rho : X \otimes \mathbb{1} \to X$ respectively called the *left and right unitors*

subject to the requirement that the diagrams

- the *triangle identity*

$$\begin{array}{ccc} (X\otimes\mathbb{1})\otimes Y & \xrightarrow{\alpha_{X,\mathbb{1},Y}} & X\otimes(\mathbb{1}\otimes Y) \\ & {\scriptstyle \rho_X\otimes\mathrm{id}_Y}\searrow \quad \swarrow{\scriptstyle \mathrm{id}_X\otimes\lambda_Y} & \\ & X\otimes Y & \end{array}$$

- and the *pentagon axiom*

$$\begin{array}{ccccc} & & (W\otimes X)\otimes(Y\otimes Z) & & \\ & {\scriptstyle \alpha_{W\otimes X,Y,Z}}\nearrow & & \searrow{\scriptstyle \alpha_{W,X,Y\otimes Z}} & \\ ((W\otimes X)\otimes Y)\otimes Z & & & & (X\otimes(X\otimes(Y\otimes Z))) \\ \downarrow{\scriptstyle \alpha_{W,X,Y}\otimes\mathrm{id}_Z} & & & & \uparrow{\scriptstyle \mathrm{id}_W\otimes\alpha_{X,Y,Z}} \\ (W\otimes(X\otimes Y))\otimes Z & & \xrightarrow{\alpha_{W,X\otimes Y,Z}} & & W\otimes((X\otimes Y)\otimes Z) \end{array}$$

commute.

If in addition the category $\mathscr{C}$ is symmetric, then there must exist

- a natural isomorphism $B_{X,Y} : X\otimes Y \to Y\otimes X$ called the *braiding*,
- which is *symmetric* in that $B_{X,Y}\circ B_{Y,X} = \mathrm{id}_{X\otimes Y}$, and
- for which the *hexagon identities*[4]

$$\begin{array}{ccccc} (X\otimes Y)\otimes Z & \xrightarrow{\alpha_{X,Y,Z}} & X\otimes(Y\otimes Z) & \xrightarrow{B_{X,Y\otimes Z}} & (Y\otimes Z)\otimes X \\ \downarrow{\scriptstyle B_{X,Y}\otimes\mathrm{id}_Z} & & & & \downarrow{\scriptstyle \alpha_{Y,Z,X}} \\ (Y\otimes X)\otimes Z & \xrightarrow{\alpha_{Y,X,Z}} & Y\otimes(X\otimes Z) & \xrightarrow{\mathrm{id}_X\otimes B_{X,Z}} & Y\otimes(Z\otimes X) \end{array}$$

and

$$\begin{array}{ccccc} X\otimes(Y\otimes Z) & \xrightarrow{\alpha^{-1}_{X,Y,Z}} & (X\otimes Y)\otimes Z & \xrightarrow{B_{X\otimes Y,Z}} & Z\otimes(X\otimes Y) \\ \downarrow{\scriptstyle \mathrm{id}_X\otimes B_{Y,Z}} & & & & \downarrow{\scriptstyle \alpha^{-1}_{Z,X,Y}} \\ X\otimes(Z\otimes Y) & \xrightarrow{\alpha^{-1}_{X,Z,Y}} & (X\otimes Z)\otimes Y & \xrightarrow{B_{X,Z}\otimes\mathrm{id}_Y} & (Z\otimes X)\otimes Y \end{array}$$

commute.

While the coherence axioms have a very natural motivation—namely to ensure that the isomorphisms between reassociated tensor products are canonical—they formally constitute a large amount of data. Moreover, coherence conditions for higher monoidal categories are themselves notoriously non-canonical and hard to

[4] Note that for a *symmetric* braiding, one of the hexagon identities (either) is redundant.

pin-down. One model[5] of these higher coherence conditions is provided by the associahedra, or Stasheff polytopes[6] [12]. This model of coherence, as with other proposals, explodes in complexity as the "categorical-level" increases. Since we intend to generalise all the way up to the ∞-level, maintaining an infinite hierarchy of coherence conditions of ever-increasing complexity will prove completely unmanageable.

It is therefore necessary that we reformulate the notion of a monoidal 1-category in order that we do not have to maintain an infinity of coherence conditions in the generalisation to the ∞-setting—ideally, they will all arrive for free.

We take inspiration from the archetypal monoidal 1-category Vec of finite dimensional vector spaces equipped with tensor product. In this category, the tensor product of a pair of vector spaces $V \otimes U$ is completely characterised by the famous universal property on its homsets; namely that for every vector space X there is bijection

$$\mathsf{Vec}(V \otimes U \to X) \cong \{f : V \times U \to X : f \text{ bilinear}\} \cong \mathsf{Vec}(V \to \mathsf{Vec}(U \to X))$$

and this bijection is natural, i.e. we require the tensor-hom adjunction hold (note that the second bijection uses the fact that Vec is a linear category). From this perspective, there is absolutely no reason why one should have equality of $V \otimes (U \otimes W)$ and $(V \otimes U) \otimes W$ (strict associativity), but nonetheless by the universal property there is a chain of natural bijections

$$\begin{aligned} \mathsf{Vec}(V \otimes (U \otimes W) \to X) &\cong \mathsf{Vec}(V \to \mathsf{Vec}(U \otimes W \to X)) \\ &\cong \mathsf{Vec}(V \to \mathsf{Vec}(U \to \mathsf{Vec}(W \to X))) \\ \mathsf{Vec}((V \otimes U) \otimes W \to X) &\cong \mathsf{Vec}(V \otimes U \to \mathsf{Vec}(W \to X)). \end{aligned}$$

Yoneda's lemma then readily convinces us that there should be an isomorphism $V \otimes (U \otimes W) \cong (V \otimes U) \otimes W$ (the centremost set in this chain consists of just the *trilinear* maps $V \times U \times W \to X$). An attempt to exploit this observation leads us to make the following definition.

Definition 2.1. Let $\mathscr{C}$ be a symmetric monoidal 1-category, and let $\underline{n} = \{1, \ldots, n\}$ and $[n] = \{0\} \cup \underline{n}$. Define a new 1-category $\mathscr{C}^{\otimes}$ with

- objects finite lists $[C_1, \ldots, C_n]$ of elements of $\mathscr{C}$, and
- a morphism $[C_1, \ldots, C_n] \to [D_1, \ldots, D_m]$ a map $\sigma : S \to \underline{m}$ for $S \subseteq \underline{n}$, along with a family of maps $f_j : \bigotimes_{i \in \sigma^{-1}\{j\}} C_i \to D_j$ indexed by $j \in \underline{m}$. By abuse of notation we sometimes call the entire data of such a morphism by $\sigma : [C_1, \ldots, C_n] \to [D_1, \ldots, D_m]$.

Note that since the monoidal 1-category $\mathscr{C}$ is symmetric, the tensor products in the definition of a morphism are defined up to canonical isomorphism. For $C \in \mathscr{C}^{\otimes}$ let $|C|$ denote the length of the finite list C.

Composition of $\sigma : [C_1, \ldots, C_n] \to [D_1, \ldots, D_m]$ and $\tau : [D_1, \ldots, D_m] \to [E_1, \ldots, E_l]$ is done by first restricting the domain of the underlying morphism $\sigma : S \to \underline{m}$ (given

[5]Another model is partially provided via Gray categories, for instance in [1], where a nontrivial diagrammatic calculus is developed.

[6]These objects are actually deeply related to the $\mathbb{E}_{\infty}^{\otimes}$ ∞-operad we will encounter below, but a description of this relationship would take us too far afield.

the underlying morphism $\tau : T \to \underline{l}$) so that $\tau \circ \sigma$ is defined, and then for each $k \in \underline{l}$ drawing the diagram

$$\bigotimes_{i \in (\tau\circ\sigma)^{-1}\{k\}} C_i \cong \bigotimes_{j \in \tau^{-1}\{k\}} \bigotimes_{i \in \sigma^{-1}\{j\}} C_i \xrightarrow{\bigotimes_{i\in\sigma^{-1}\{j\}} f_i} \bigotimes_{k \in \tau^{-1}\{j\}} D_j \xrightarrow{g_k} E_k$$

and defining $h_k : \bigotimes_{i \in (\tau\circ\sigma)^{-1}\{k\}} C_i \to E_k$ to be the composite it specifies.

The slightly clumsy way in which we had to formulate the notion of morphisms in Definition 2.1 gives cause to make an additional definition, which at the moment might seem to exist only for the superficial purpose of bookkeeping. Let Fin_* be a skeleton of the category of finite pointed sets, namely with objects $[n]$ for any $n \in \mathbf{N}_0$, and morphisms maps $[n] \to [m]$ for which $0 \mapsto 0$ (that this category is equivalent to the category of all pointed sets and pointed maps is obvious). Then the map $\sigma : S \to [|C'|]$ associated to a morphism $C \to C'$ in $\mathscr{C}^\otimes$ is exactly the data of a map $\sigma' \in \mathsf{Fin}_*([|C|] \to [|C'|])$, where we define $\sigma' = \sigma$ on S, and $\sigma' = 0$ (the point) on $[|C|] \setminus S$. We will freely switch perspectives as it proves convenient.

Given a symmetric monoidal 1-category $\mathscr{C}$, there is an evident forgetful functor $p : \mathscr{C}^\otimes \to \mathsf{Fin}_*$ defined on objects by $C \mapsto [|C|]$ (i.e. $[C_1, \dots, C_n] \mapsto [n]$). We will spend the rest of this section establishing that from $\mathscr{C}^\otimes$ along with $p : \mathscr{C}^\otimes \to \mathsf{Fin}_*$ all of the data of the symmetric monoidal 1-category $\mathscr{C}$ can be recovered, and in addition we will obtain conditions under which a map $p : \mathscr{D} \to \mathsf{Fin}_*$ for $\mathscr{D}$ a 1-category gives rise to a symmetric monoidal category. Our first proposition establishes that at least the underlying 1-category of $\mathscr{C}$ is preserved by the construction of Definition 2.1, being the fibre of p over $[1]$.

Lemma 2.2 (Warm-up)**.** *The fibre $\mathscr{C}^\otimes_{[n]}$ of p over $[n]$ is a category, and moreover there is an equivalence of categories $\mathscr{C}^\otimes_{[1]} \to \mathscr{C}$.*

Proof. The first part follows directly from the definition. For the second, let $[C] \in \mathscr{C}^\otimes_{[1]}$ and define a functor $F : \mathscr{C}^\otimes_{[1]} \to \mathscr{C}$ on objects by $F([C]) = C$. Now let $\sigma : [C] \to [D]$ be a morphism in $\mathscr{C}^\otimes$, with underlying morphism $\sigma : S \to \underline{1} = \{1\}$ (and $S \subseteq \{1\}$). Since σ lies in the fibre of p over $[1]$ we must have $p(\sigma) = \mathrm{id}_{[1]}$, and hence $S = \{1\}$ so σ comes along with the data of a map $f_1 : C \to D$. Thus we can define $F(\sigma) = f_1$. That this is a functor is immediate from the definition of a morphism in $\mathscr{C}^\otimes$, and F thus constructed is obviously fully faithful. Moreover F is surjective on objects, so $F : \mathscr{C}^\otimes_{[1]} \to \mathscr{C}$ is an isomorphism. □

This proposition concretely explains the (simple) way in which the underlying category of a symmetric monoidal tensor category can be recovered from the category $\mathscr{C}^\otimes$. It is our next objective to see that the symmetric monoidal structure can be recovered, too, and in fact this will follow from a general fact that any map $q : \mathscr{D} \to \mathsf{Fin}_*$ whatever satisfying a pair of conditions gives rise to a symmetric monoidal category. We first realise these properties for our concrete $\mathscr{C}^\otimes$ construction as a pair of propositions.

Proposition 2.3 (Property 1)**.** *The map $p : \mathscr{C}^\otimes \to Fin_*$ is an op-fibration. That is, for any $C \in \mathscr{C}^\otimes$ and map $\psi \in Fin_*([|C|] \to [n])$, there is a map $\tilde{\psi} : C \to C'$ p-covering ψ (i.e. with $|C'| = n$) with the universal property that for any $C'' \in \mathscr{C}^\otimes$ the map*

$$\mathscr{C}^\otimes(C' \to C'') \xrightarrow{\tilde{\psi}\circ -} \mathscr{C}^\otimes(C \to C'') \times_{Fin_*([|C|] \to [|C''|])} Fin_*([|C'|] \to [|C''|])$$

induced by precomposition by $\tilde{\psi}$ is a bijection (the fibre product of the codomain is taken over the pair of maps $(p, \psi \circ -)$, the latter being precomposition with f).

Proof. Let $C \in \mathscr{C}^{\otimes}$ and $\psi \in \mathsf{Fin}_*([|C|] \to [n])$, and fix a morphism $\tau \in \mathscr{C}^{\otimes}(C' \to C'')$ with $|C'| = n$ and underlying map $\tau \in \mathsf{Fin}_*([|C'|] \to [|C''|])$ of the same name. Let the family of $\underline{|C''|}$-indexed morphisms associated to τ be labelled g_k. Then we get a map $\sigma : [|C|] \to [|C''|]$ by setting $\sigma = \tau \circ \psi$. For each $k \in \underline{|C''|}$ we now need to define

$$h_k : \bigotimes_{i \in \sigma^{-1}\{k\}} C_i \cong \bigotimes_{j \in \tau^{-1}\{k\}} \bigotimes_{i \in f^{-1}\{j\}} C_i \longrightarrow \bigotimes_{k \in \tau^{-1}\{j\}} C'_j \xrightarrow{g_k} C''_k,$$

where again the first isomorphism comes from the fact that $\mathscr{C}$ is symmetric monoidal. We now see that there is essentially "only one" way to define $\tilde{\psi} : C \to C'$ (and thus C'); namely by having an associated morphism $\psi : [|C|] \to [|C'|]$ and family of morphisms (for $j \in \underline{|C'|}$)

$$f_j : \bigotimes_{i \in \psi^{-1}\{j\}} C_i \to C'_j = \bigotimes_{i \in \psi^{-1}\{j\}} C_i, \quad \text{i.e. with} \quad C'_j = \bigotimes_{i \in \psi^{-1}\{j\}} C_i.$$

Of course, this choice of C' and morphism $\tilde{\psi}$ must not depend on C'' or τ, but this is obviously the case.

Given this definition of $\tilde{\psi}$ we thus obtain a morphism

$$\sigma \in \mathscr{C}^{\otimes}(C \to C'') \times_{\mathsf{Fin}_*([|C|] \to [|C''|])} \mathsf{Fin}_*([|C'|] \to [|C''|]),$$

which in particular lies in the fibre product since $p(\sigma) = \tau \circ \psi$ and hence factors through ψ. On the other hand if $\sigma \in \mathscr{C}^{\otimes}(C \to C'')$ has $p(\sigma)$ factoring through ψ then we obtain the desired morphism $\psi \in \mathscr{C}^{\otimes}(C' \to C'')$ just by definition of what it means to be a morphism in $\mathscr{C}^{\otimes}$. These constructions are clearly inverses, so we obtain the desired bijection. □

Considering the op-fibration p abstractly (forgetting about the definition of $\mathscr{C}^{\otimes}$), this lemma almost gives a recipe for how to tensor together two objects $C_1, C_2 \in \mathscr{C}$. Namely we define $\psi : [2] \to [1] \in \mathsf{Fin}_*$ by specifying $\psi(1) = \psi(2) = 1$, since then after forming $C = [C_1, C_2] \in \mathscr{C}^{\otimes}$ the object $C' = [C'] \in \mathscr{C}^{\otimes}$ provided by the lemma is exactly the tensor product $C_1 \otimes C_2$, and the map $\tilde{\psi}$ explains how to compare the "formal tensor product" $[C_1, C_2] \in \mathscr{C}^{\otimes}$ with the actual one $[C_1 \otimes C_2]$. However, we have been slightly dishonest here; we have still used our knowledge that the elements of $\mathscr{C}^{\otimes}$ are lists in the objects of the underlying category $\mathscr{C}$.

For a generic morphism $q : \mathscr{D} \to \mathsf{Fin}_*$ satisfying Proposition 2.3 we unfortunately lack a way to interpret arbitrary elements of $\mathscr{D}$ as lists. The second key property of $p : \mathscr{C}^{\otimes} \to \mathsf{Fin}_*$ generalises Lemma 2.2 in order to deal with this problem.

Proposition 2.4 (Property 2)**.** *The fibres $\mathscr{C}^{\otimes}_{[n]}$ (which we have seen are categories) are each respectively equivalent to the Cartesian product $(\mathscr{C}_{[1]})^n$ for each $n \geq 0$. The projection of this equivalence onto the ith component of the product is induced by applying Proposition* 2.3 *to the map*

$$\rho_{n,i} : [n] \to [1], j \mapsto \delta_{i,j}.$$

Since we have seen that $\mathscr{C}^{\otimes}_{[1]} \cong \mathscr{C}$, this means that $\mathscr{C}^{\otimes}_{[n]} \cong \mathscr{C}^n$ (we have rediscovered our finite lists).

Proof. The $n = 0$ case is special; by definition $[]$ is the only object of $\mathscr{C}^{\otimes}_{[0]}$, and therefore since $\mathscr{C}^{\otimes}_{[0]}$ has a unique morphism it is isomorphic to the terminal category 1.

For $n \geq 1$ we first establish that the maps $\rho_{n,i}$ give rise to functors $\rho^{!}_{n,i} : \mathscr{C}^{\otimes}_{[n]} \to \mathscr{C}^{\otimes}_{[1]}$. Given an object $C = [C_1, \ldots, C_n] \in \mathscr{C}^{\otimes}_{[n]}$ Proposition 2.3 immediately yields an object $[D] \in \mathscr{C}^{\otimes}_{[1]}$ and a morphism $\tilde{\psi} : C \to [D] \in \mathscr{C}^{\otimes}$ covering $\rho_{n,i}$ and satisfying the claimed universal property. We then define $\rho^{!}_{n,i}(C) = [D]$.

Now let $\sigma : C \to C'$ be a morphism in $\mathscr{C}^{\otimes}_{[n]}$. Then as just described we have a diagram

$$\begin{array}{ccc} C & \xrightarrow{\ \sigma\ } & C' \\ {\scriptstyle \tilde{\psi}}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\psi}'} \\ [D] & \dashrightarrow & [D'] \end{array} .$$

Since the composite $\tilde{\psi}' \circ \sigma$ projects to $p(\tilde{\psi}' \circ \sigma) = \rho_{n,i} \circ p(\sigma)$, a map which factors through $\rho_{n,i}$, by the universal property of $\tilde{\psi}$ there exists a unique $\hat{\sigma} : [D] \to [D']$ fitting into the diagram as the dotted morphism. Under p commutativity of the diagram yields $p(\hat{\sigma}) \circ \rho_{n,i} = \rho_{n,i} \circ p(\sigma) = \rho_{n,i}$ (since σ is assumed to lie in $\mathscr{C}^{\otimes}_{[n]}$ and hence p-covers $\mathrm{id}_{[n]}$). Therefore $p(\hat{\sigma}) = \mathrm{id}_{[1]}$, which shows that $\hat{\sigma}$ is a morphism in $\mathscr{C}^{\otimes}_{[1]}$ and we define $\rho^{!}_{n,i}(\sigma) = \hat{\sigma}$. The universal property provided by Proposition 2.3 ensure that $\rho^{!}_{n,i}$ as constructed is actually a functor.

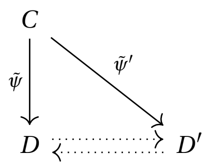
Together these functors $\rho^{!}_{n,i}$ assemble in the obvious way into a functor $\rho^{!}_{n} : \mathscr{C}^{\otimes}_{[n]} \to \mathscr{C}^{n}$. Unravelling definitions this functor sends $[C_1, \ldots, C_n]$ to $(C_1, \ldots, C_n)$. Moreover the data of a morphism $\sigma : [C_1, \ldots, C_n] \to [C'_1, \ldots, C'_n]$ *which covers* $\mathrm{id}_{[n]}$ is the data of a map $f_i : C_i \to C'_i$ in $\mathscr{C}$ for each $i \in \underline{n}$, and as it happens by construction that $\rho^{!}_{n}(\sigma) = (f_1, \ldots, f_n)$. This functor is obviously fully faithful and surjective, so is an isomorphism. □

Observe that in the proof the the lemma we only needed information about the definition of $\mathscr{C}^{\otimes}$, with the exception of that provided by Proposition 2.3, in order to verify that we had an equivalence. This is a symptom of the fact that henceforth we need not pay attention to the actual definition of $\mathscr{C}^{\otimes}$ in showing that we may extract the data of a symmetric monoidal category from it; we will only need the content of Propositions 2.3 and 2.4 together. Nonetheless we will actually refer back to the fact that the objects of $\mathscr{C}^{\otimes}$ are lists in order to see that we actually recover the *same* symmetric monoidal category $\mathscr{C}$ that we started with. This is the purpose of the next proposition, but we first need an additional lemma.

Lemma 2.5. *Assume $p : \mathscr{D} \to Fin_*$ is a functor for which Proposition* 2.3 *holds. Then every morphism $\psi : [n] \to [m]$ in Fin_* determines a functor $\psi^{!} : \mathscr{D}_{[n]} \to \mathscr{D}_{[m]}$ up to canonical isomorphism. This is just a generalisation of the maps $\rho^{!}_{n,i}$ of Proposition* 2.4.

Proof. We have already seen how to construct the functor $\psi^{!}$ in the proof of Proposition 2.4. The "up to canonical isomorphism" part just comes from the fact that given $C \in \mathscr{D}$ and $\psi \in \mathsf{Fin}_*([|C|] \to [n])$ the object D (along with the morphism $C \to D$) provided by Proposition 2.3 is determined only up to canonical isomorphism (and then the generic way in which a functor is determined by a universal property). Indeed if $\tilde{\psi} : C \to D$ and $\tilde{\psi}' : C \to D'$ are both possible morphisms returned by Proposition 2.3

then we can form the diagram



and because $p(\tilde{\psi}) = p(\tilde{\psi}') = \psi$ by hypothesis, the universal properties of these morphisms provide unique morphisms $\sigma : D \to D'$ and $\sigma : D \to D'$ both fitting into this diagram as dotted morphisms and making it commute. Using the universal properties again we find that σ and σ' are mutually inverse and therefore $D \cong D'$, as desired. □

Proposition 2.6. *A functor $p : \mathscr{D} \to \mathsf{Fin}_*$ for which Propositions* 2.3 *and* 2.4 *both hold defines a symmetric monoidal category. If $\mathscr{D} = \mathscr{C}^{\otimes}$ for a symmetric monoidal category $\mathscr{C}$ then we recover $\mathscr{C}$ up to equivalence.*

Proof. We will not bother to check all of the axioms of a symmetric monoidal category here—once all of the necessary data is pointed out this becomes an elementary exercise, and is precisely the advantage of the reformulation.

Certainly the fibre $\mathscr{D}_{[1]}$ is a category, and it will be the underlying category upon which we will construct a symmetric monoidal structure. We now make ample use of Lemma 2.5. It first turns the unique pointed map $0 : [0] \to [1]$ into a functor $\mathscr{D}_{[0]} \to \mathscr{D}_{[1]}$. Proposition 2.4 gives that $\mathscr{D}_{[0]}$ has a single isomorphism class, and so we can take an object in the image to be the tensor unit $\mathbb{1}$ (defined up to isomorphism).

As we saw in the concrete case of $\mathscr{D} = \mathscr{C}^{\otimes}$ after the proof of Proposition 2.3, the monoidal product $\otimes : \mathscr{D}_{[1]} \times \mathscr{D}_{[1]} \to \mathscr{D}_{[1]}$ is obtained by the composite $(\mathscr{D}_{[1]})^2 \cong \mathscr{D}_{[2]} \to \mathscr{D}_{[1]}$ with the latter map induced by Lemma 2.5 applied to $\psi : [2] \to [1]$ with $\psi(1) = \psi(2) = 1$ (this determines the result of a tensor product up to canonical isomorphism). The braiding is obtained in exactly the same way from $\phi : [2] \to [2]$ defined by $\phi(1) = 2$ and $\phi(2) = 1$. Moreover, since precomposition of ψ with ϕ leaves ψ invariant, by functoriality this braiding must be symmetric.

The associator is obtained in a way which is only slightly more subtle. There are two ways to construct a map $[3] \to [1]$ in Fin_* which collapses $\{1,2,3\}$ to $1 \in [1]$, given that we only collapse together adjacent pairs of elements of [3] one-at-a-time;

1. One way is to define $[3] \to [2]$ by sending $\{1,2\}$ to 1 and 3 to 2, and then using $[2] \to [1]$ which sends $\{1,2\}$ to 1.

2. The other way is to collapse $\{2,3\}$ first by sending both to 2 in the definition of the map $[3] \to [2]$, fixing 0 (obviously) and 1, followed again by using $[2] \to [1]$ which sends $\{1,2\}$ to 1.

In either case, the resulting composite map $[3] \to [1]$ is the same. By Lemma 2.5 each way to construct this map picks out the same functor up to canonical isomorphism. Way (1) represents the process

$$[A,B,C] \mapsto [A \otimes B, C] \mapsto [(A \otimes B) \otimes C],$$

while way (2) represents the process

$$[A,B,C] \mapsto [A, B \otimes C] \mapsto [A \otimes (B \otimes C)].$$

The canonical isomorphism between the functors thus induced is exactly the associator $\alpha_{A,B,C} : (A\otimes B)\otimes C \to A\otimes(B\otimes C)$ of the monoidal category. The triangle identity and unitors can be obtained in exactly the same fashion; each unitor is the functor induced by applying Lemma 2.5 to the map $[2] \to [1]$ defined by sending either 1 or 2 to 1, and the other of these to 0. The triangle identity then just amounts to the (evident) commutativity of a representing diagram in Fin_*. Translating the pentagon axiom into a diagram of representing morphisms in Fin_* also immediately yields its commutativity for the same reason. Finally, the hexagon identities are proved in the analogous way, exploiting the map $\phi : [2] \to [2]$ which swaps 1 and 2—which we used above to show symmetry of the braiding.

In the special case of $\mathscr{D} = \mathscr{C}^{\otimes}$ unravelling definitions recovers all of the original symmetric monoidal structure up to canonical isomorphism. The only case in which this is perhaps slightly subtle is that of the braiding, which was not explicitly mentioned in the definition of $\mathscr{C}^{\otimes}$. Nonetheless it is actually built into the definition of composites of morphisms in $\mathscr{C}^{\otimes}$, since it is used to obtain a canonical isomorphism

$$\bigotimes_{i\in(\tau\circ\sigma)^{-1}\{k\}} C_i \cong \bigotimes_{j\in\tau^{-1}\{k\}} \bigotimes_{i\in\sigma^{-1}\{j\}} C_i .$$

□

Thus the pentagon axiom is reduced to a triviality. Further, in hoping to generalise to n-categories it was actually the least of our worries, since in the ∞-case we would have an infinite hierarchy of coherence conditions each rapidly increasing in complexity.

We are finally ready to posit the definition of a symmetric monoidal ∞-category; we conclude this section by recalling the analogous version of an op-fibration for quasicategories, followed by a direct translation of the ordinary symmetric monoidal category reformulation into the ∞-setting.

Definition 2.7. Fix an inner fibration $p : C \to D$ of simplicial sets. An edge $f : x \to y$ of a simplicial set C is said to be a *p-Cartesian* edge if the induced map

$$C_{/f} \to C_{/y} \times_{D/p(y)} D_{/p(f)}$$

is a trivial Kan fibration.

The original map $p : C \to D$ is then said to be a *Cartesian fibration* of simplicial sets if in addition to it being an inner fibration, it satisfies the following condition: for every edge $f : x \to y \in D_1$ and vertex $\tilde{y} \in C_0$ lifting $y \in D_0$ there exists a p-Cartesian edge $\tilde{f} : \tilde{x} \to \tilde{y} \in C_1$ lifting all of f. If $p^{\mathrm{op}} : C^{\mathrm{op}} \to D^{\mathrm{op}}$ is a *Cartesian fibration* then the original map p is said to be an *opCartesian* or *coCartesian fibration*.

Definition 2.8. A *symmetric monoidal ∞-category* is an opCartesian fibration of simplicial sets

$$p : \mathscr{C}^{\otimes} \to N\mathsf{Fin}_*$$

such that the functors $\rho^!_{n,i} : \mathscr{C}^{\otimes}_{[n]} \to \mathscr{C}^{\otimes}_{[1]}$ induced by each $\rho_{n,i} : [n] \to [1] \in \mathsf{Fin}_*$ assemble together into an equivalence $\mathscr{C}^{\otimes}_{[n]} \cong (\mathscr{C}^{\otimes}_{[1]})^n$. We will call this latter condition $(\star)$.

The "op-fibration" condition translates Proposition 2.3, while the "Cartesianness" requirement is the direct analogy of Lemma 2.5. The additional condition on p is the conclusion of Proposition 2.4 verbatim.

The point of the notation $\mathscr{C}^\otimes$ is to distinguish the simplicial set $\mathscr{C}^\otimes$ from its fibre $\mathscr{C} = \mathscr{C}^\otimes_{[1]}$ (which is necessarily a quasicategory), and which in analogy with the 1-category case we call the "underlying quasicategory" of $\mathscr{C}^\otimes$.

3 The general formalism

We now manoeuvre to interpret Definition 2.8 from the perspective of the general formalism. Let $p : \mathscr{C}^\otimes \to N\mathsf{Fin}_*$ be a symmetric monoidal ∞-category. The idea is to think of this data actually as a composite

$$\mathscr{C}^\otimes \xrightarrow{\quad p \quad} \mathscr{O}^\otimes = N\mathsf{Fin}_* \xrightarrow{q=\mathrm{id}_{N\mathsf{Fin}_*}} N\mathsf{Fin}_*$$

i.e. of p with the identity, which together satisfies some conditions.

First consider the latter map $q : \mathscr{O}^\otimes = N\mathsf{Fin}_* \to N\mathsf{Fin}_*$. Since it is the identity, we can think of this map as satisfying condition ($\star$), and in addition possessing a weakened version of opCartesian-ness in order that ($\star$) makes sense for a non-opCartesian map. A morphism of simplicial sets having both of these properties is called an ∞-operad, generalising the categorical notation of a coloured operad (we will see how soon). More precisely, we have the following:

Definition 3.1. A morphism $\psi \in \mathsf{Fin}_*([n] \to [m])$ is *inert* if it is surjective, and in addition, is injective off the fibre $\psi^{-1}\{0\}$ of the distinguished point.

An ∞-*operad* is a map $q : \mathscr{O}^\otimes \to N\mathsf{Fin}_*$ of quasicategories for which the following three conditions hold.

1. The "functor induced by an inert morphism" makes sense—For every inert map $\psi \in \mathsf{Fin}_*([n] \to [m])$ and object $C \in \mathscr{O}^\otimes_{[n]}$ there exists a q-opCartesian edge $\tilde{\psi} : C \to C'$ which q-covers ψ, so that in particular we obtain a functor $\psi^! : \mathscr{O}^\otimes_{[n]} \to \mathscr{O}^\otimes_{[m]}$ (as we had in the general opCartesian case).

2. There is a coherently associative multiplication of operations in some sense—Let $f \in \mathsf{Fin}_*([n] \to [m])$ and objects $C \in \mathscr{C}^\otimes_{[n]}$ and $C' \in \mathscr{C}^\otimes_{[m]}$ be arbitrary, and let $\mathrm{Map}^f_{\mathscr{O}^\otimes}(C, C')$ denote the union of the connected components of $\mathrm{Map}_{\mathscr{O}^\otimes}(C, C')$ which q-cover f. Then for every family $\psi_i : C' \to C'_i$ of morphisms p-covering each respective $\rho_{n,i}$ the evident map

$$\mathrm{Map}^f_{\mathscr{O}^\otimes}(C, C') \to \prod_{i \in \underline{n}} \mathrm{Map}^{\psi_i \circ f}_{\mathscr{O}^\otimes}(C, C'_i)$$

is a homotopy equivalence. This is a technical condition which we will not pay much attention to later.

3. Condition ($\star$)—The maps $\rho_{n,i} : [n] \to [1]$ induce equivalences $\rho^!_n : \mathscr{O}^\otimes_{[n]} \cong (\mathscr{O}^\otimes_{[1]})^n$ for each $n \geq 0$.

The other piece of the composite $p : \mathscr{C}^\otimes \to \mathscr{O}^\otimes$ is then required to be an opCartesian fibration, for which the we also have condition ($\star$). Since an opCartesian fibration automatically has the weaker properties (1) and (2) above required to be an ∞-operad, another way to say this is that $p : \mathscr{C}^\otimes \to \mathscr{O}^\otimes$ is an opCartesian fibration for which the composite $q \circ p$ is an ∞-operad. Moreover, this makes sense for arbitrary ∞-operads $\mathscr{O}^\otimes$, and leads to the following definition:

Definition 3.2. Let $q : \mathscr{O}^{\otimes} \to N\mathsf{Fin}_*$ be an ∞-operad. An *$\mathscr{O}^{\otimes}$-monoidal ∞-category* is a coCartesian fibration $p : \mathscr{C}^{\otimes} \to \mathscr{O}^{\otimes}$ for which $q \circ p$ is an ∞-operad.

From this perspective of greater generality, the subtlety in our original development arises since by "coincidence" in this case $\mathscr{O}^{\otimes}$ and $N\mathsf{Fin}_*$ actually happen to be the same. To help distinguish when $N\mathsf{Fin}_*$ is to be thought of as an ∞-operad or just the target of an ∞-functor, we define $\mathrm{Comm}^{\otimes} = N\mathsf{Fin}_*$ to be the *commutative ∞-operad* (coming from the ordinary *commutative operad* **Comm** with $\mathbf{Comm}^{\otimes} = \mathsf{Fin}_*$). This is inspired by the fact that $\mathrm{Comm}^{\otimes}$-monoidal ∞-categories are exactly the symmetric monoidal ∞-categories, as we have have just seen directly.

In contrast with the perspective of ordinary 1-category theory, we have not been thinking of a *symmetric* monoidal ∞-category as simply a monoidal ∞-category with additional structure (indeed, we do not even yet know what the latter thing is meant to mean). In fact, in translating from the 1-category world to the ∞-world we could have just began with an ordinary monoidal category $\mathscr{C}$ and constructed $\mathscr{C}^{\otimes}$ in almost the same way; letting the morphisms be weakly order preserving maps, and then equipping the construction with a forgetful functor to $\mathbf{\Delta}^{\mathrm{op}}$ (as opposed to Fin_*). The result would have been a notion of an ordinary monoidal ∞-category, but there are practical disadvantages associated with this perspective (c.f. [8] with [7]).

It is better to realise ordinary monoidal ∞-categories as $\mathscr{O}^{\otimes}$-monoidal infinity categories over just some other ∞-operad $\mathscr{O}^{\otimes}$. This is most easily done via a generalisation of the construction of $\mathscr{C}^{\otimes}$ for symmetric monoidal 1-categories given above to the setting of coloured operads. We will not give the precise definition of a coloured operad here (see [6] for a succinct introduction), but intuitively a coloured operad can be thought of as an ordinary category $\mathscr{C}$ equipped not only with homsets $\mathscr{C}(C \to C')$ but so-called "multiplications" $\mathscr{C}([C_1, \dots, C_n] \to C')$ permitting finite lists in the first argument. In addition the "multiplication operations"—the elements of these sets—should possess a notion of "insertion", taking an n-ary operation and n other k_i-ary operations yielding a single $(\sum_i k_i)$-ary operation, and this should satisfy some set of coherence conditions. As in the symmetric monoidal case these coherence conditions rapidly become unmanageable in the transition to higher categories, but by performing a slight generalisation of the construction of Definition 2.1 these issues are similarly resolved. The reformulation essentially amounts to unbreaking the asymmetry in the definition of the sets $\mathscr{C}([C_1, \dots, C_n] \to C')$ by permitting a finite list in the second argument as well.

An ordinary operad is a coloured operad with a single object. We will define an operad **Assoc** (with the goal of characterising associativity) with object $\bullet$ by letting $\mathscr{C}([\bullet]_{i \in \underline{n}} \to \bullet)$ be the set of linear orders on $\underline{n}$. Essentially the only remaining data we must provide is how to insert n k_i-ary operations (i.e. linear orders $\leq_i \in \mathscr{C}([\bullet]_{j \in \underline{k_i}} \to \bullet)$ for $1 \leq i \leq n$) into an element $\leq \in \mathscr{C}([\bullet]_{i \in \underline{n}} \to \bullet)$. Formally given a map $\psi : \underline{K} \to \underline{n}$ with $k_i = |\psi^{-1}\{i\}|$ we produce a linear order on $\underline{K}$ by forming the associated *block ordering*; for $k, k' \in \underline{K}$ we set $k \leq k'$ whenever $\psi(k) < \psi(k')$ (between blocks) or $\psi(k) = \psi(k')$ (within a block) and setting $S = \psi^{-1}\{\psi(k)\}$ we have

$$|S \cap \underline{k}| \leq_i |S \cap \underline{k'}|.$$

By the minor generalisation of the Definition 2.1 construction of Lurie[7] we obtain an ordinary category $\mathbf{Assoc}^{\otimes}$ along with a forgetful functor to Fin_*. Upon taking nerves this immediately gives rise to the data of an ∞-operad $\mathrm{Assoc}^{\otimes} = N\mathbf{Assoc}^{\otimes} \to N\mathsf{Fin}_*$ called the *associative ∞-operad*. We then make the following definition.

[7] Construction 2.1.1.7 of [10]

Definition 3.3. A *monoidal* ∞*-category* is an $\mathrm{Assoc}^{\otimes}$-monoidal ∞-category.

Since $N\mathbf{\Delta}^{\mathrm{op}}$ is not an ∞-operad, the equivalence of the notions of $\mathrm{Assoc}^{\otimes}$-monoidal ∞-categories and monoidal ∞-categories obtained from forgetful functors $\mathscr{C}^{\otimes} \to N\mathbf{\Delta}^{\mathrm{op}}$ must be made precise in another sense. The correct sense is that both constructions give rise to an equivalent category of associated *algebra objects*.

It is interesting to see how ∞-operads give rise to algebra objects in general. We first consider the case of an algebra over the ordinary operad **Assoc**, which is the data of a coloured operad map $\iota : \mathbf{Assoc} \to \mathscr{C}$ for $\mathscr{C}$ a symmetric monoidal 1-category. In particular, the map ι picks out for us an object C of the category $\mathscr{C}$ determined by where its unique object $\bullet$ is sent. By the definition of a coloured operad map ι must also provide a way for translating elements of $\mathrm{Mul}_{\mathbf{Assoc}}([\bullet]_{i\in\underline{n}}, \bullet)$, i.e. linear orders on $\underline{m}$, into morphisms

$$\underbrace{C \otimes \cdots \otimes C}_{n\text{-times}} \to C = \iota(\bullet)$$

in $\mathscr{C}$. From these "products" on C the data of an associative algebra object of the symmetric monoidal category $\mathscr{C}$ (of which we give the *symmetric* variant in the introduction) is readily recovered. In particular the morphism associated to the empty linear order on the empty set gives the morphism $\mathbb{1} \to C$ in $\mathscr{C}$ picking out the tensor unit, and a product $C \otimes C \to C$ is recovered from the standard linear order on $\underline{2}$. The compatibility conditions which come along with a map of coloured operads (using the "multiplication product" in the operad **Assoc** to associate $A \otimes A \otimes A$ in the two possible ways) ensure that this induced product morphism $C \otimes C \to C$ in $\mathscr{C}$ is associative. We have confidence simply calling operatic maps $\mathbf{Assoc} \to \mathscr{C}$ in this way associative algebras on their own, since it is easy to construct such a map from the data of an arbitrary associative algebra object of $\mathscr{C}$ (each of the linear orders of $\mathrm{Mul}_{\mathbf{Assoc}}([\bullet]_{i\in\underline{n}}, \bullet)$ give instructions for performing a reassociation via the associative algebra object structure).

Although lying completely within in the 1-categorical setting, this example is illustrative of the general features of algebras over ∞-operads. We have seen that really, two ∞-operads should play a role:

- There is an ∞-operad $\mathscr{O}^{\otimes}$ which *controls* the theory of algebras, in that an algebra is an ∞-functor $\iota : \mathscr{O}^{\otimes} \to \mathscr{C}^{\otimes}$.

- There is a second ∞-operad $\mathscr{O}'^{\otimes}$ which is required to specify the kind of monoidal ∞-category which $\mathscr{C}^{\otimes}$ is (i.e. $\mathscr{C}^{\otimes}$ should be $\mathscr{O}'^{\otimes}$-monoidal).

In the 1-categorical setting we considered above we investigated the analogous situation of **Assoc**-algebras in the **Comm**-monoidal 1-category $\mathscr{C}$.

Since 1-categorical algebras were just "operadic" maps of operads, this motivates the following pair of definitions.

Definition 3.4. A functor $f : \mathscr{O}^{\otimes} \to \mathscr{O}'^{\otimes}$ between ∞-operads $p : \mathscr{O}^{\otimes} \to N\mathsf{Fin}_*$ and $q : \mathscr{O}'^{\otimes} \to N\mathsf{Fin}_*$ is a *map of* ∞*-operads* if it preserves inert morphisms and the diagram

$$\begin{array}{ccc} \mathscr{O}^{\otimes} & \xrightarrow{\quad f \quad} & \mathscr{O}'^{\otimes} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & N\mathsf{Fin}_* & \end{array}$$

commutes.

An edge f of an ∞-operad $p : \mathcal{O}^{\otimes} \to N\mathsf{Fin}_*$ is *inert* if it is opCartesian and $p(f)$ is inert. This technical condition on inert morphisms is needed so (and is in fact equivalent to asking) that inert morphisms covering the maps $\rho_{n,i}$ are sent to inert morphisms. Thus the functor is f is made compatible with the ∞-operadic structure.

Definition 3.5. Let $p : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ be a map of ∞-operads and let $q : \mathcal{C}^{\otimes} \to \mathcal{O}'^{\otimes}$ be an $\mathcal{O}'^{\otimes}$-monoidal ∞-category (which itself necessarily also becomes an ∞-operad as we have seen). Then a map of ∞-operads $f : \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ is a *$\mathcal{O}^{\otimes}$-algebra in $\mathcal{C}^{\otimes}$* if the diagram

$$\begin{array}{ccc} \mathcal{O}^{\otimes} & \xrightarrow{\quad f \quad} & \mathcal{C}^{\otimes} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathcal{O}'^{\otimes} & \end{array}$$

commutes. The full sub-∞-category of the $\mathcal{O}'^{\otimes}$-slice mapping space which consists of $\mathcal{O}'^{\otimes}$-algebras in $\mathcal{C}^{\otimes}$ is denoted $\mathbf{Alg}_{\mathcal{O}/\mathcal{O}'}(\mathcal{C})$ (by abuse of notation). In the special case of $\mathcal{O}'^{\otimes} = \mathrm{Comm}^{\otimes}$ (so $\mathcal{C}^{\otimes}$ is symmetric monoidal) we use the shorthand $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C})$, and when additionally $\mathcal{O}' = \mathrm{Comm}^{\otimes}$ we write $\mathbf{CAlg}(\mathcal{C})$ for the same, which we call the *∞-category of commutative algebra objects of $\mathcal{C}$*.

This is how generalised algebras are constructed in the ∞-category formalism. It turns out that a theory of modules associated to these algebras is more elusive, and cannot be realised at the same level of generality. Instead we must restrict to the situation of *coherent*[8] ∞-operads, where some technical conditions (which we will avoid specifying) are asserted which "guarantee the existence of a reasonable theory of tensor products for modules over their algebras". In this situation, however, we get associated ∞-operads of modules with the resulting theory of commutative algebra begin very robust (we will discuss this below).

Given the definition of the commutative algebra objects of a symmetric monoidal ∞-category, we are now able to elaborate on the sense in which $\mathrm{Assoc}^{\otimes}$-monoidal ∞-categories, and those monoidal ∞-categories constructed via a(n unconventional) forgetful functor to $N\mathbf{\Delta}^{\mathrm{op}}$, are related. The idea is that an arbitrary ∞-category $\mathcal{C}$ which admits finite products can be equipped with an essentially unique Cartesian symmetric monoidal structure[9] (see Subsection 2.4.1 of [10]). The objects of $\mathbf{CAlg}(\mathcal{C})$ are in this case called *monoid objects* of $\mathcal{C}$. Alternatively, the notion of a monoid may be defined directly by specifying a particular[10] kind of map of ∞-categories $N\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$, with the ∞-operad of all such objects called $\mathbf{Mon}(\mathcal{C})$. It is then a theorem there is a functor $\mathrm{Cut} : N\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Assoc}^{\otimes}$ induced by *taking cuts*[11] induces an equivalence of the respective associated ∞-categories $\mathbf{CAlg}(\mathcal{C})$ and $\mathbf{Mon}(\mathcal{C})$ of algebra objects.

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[8] A precise definition and the beginning of the development of theory of associated modules can be found in Subsection 3.3.1 of [10].

[9] Incidentally, this same construction confers the ∞-category of ∞-categories Cat_∞ a symmetric monoidal structure. Then, for example, the commutative $\mathbb{E}_\infty^{\otimes}$-algebra objects of Cat_∞ are exactly the symmetric monoidal ∞-categories.

[10] Definition 4.1.2.5. of [10].

[11] The functor is easily defined by identifying each element i of $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ with the partitioning of $[n]$ into the pair of disjoint subsets of elements strictly less than, and then greater than or equal to, i.

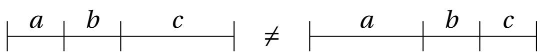
The associative and commutative ∞-operads which we have considered here represent opposite poles of an infinite sequence of ∞-operads, the operads $\mathbb{E}_k^\otimes$, which generalise associative algebras. By definition $\mathbb{E}_\infty^\otimes = \mathrm{Comm}^\otimes$, while (we will see that) the ∞-operads $\mathbb{E}_1^\otimes$ and $\mathrm{Assoc}^\otimes$ are equivalent. Introducing the general framework into which these ∞-operads fit, and hence the motivation for these names, is the purpose of the next section. In short, the $\mathbb{E}_k^\otimes$ ∞-operads arise from the (more) classical *little k-cubes* operads of Boardman and Vogt [2, 13, 9].

4 The rest of the spectrum; little k-cubes

We begin with some topology; let $X = (X, x_0) \in \mathsf{Top}_*$ be a pointed space. In this setting there is the classical *loop space* functor $\Omega : \mathsf{Top}_* \to \mathsf{Top}_*$ defined by $X \mapsto \mathsf{Top}_*(S^1 \to X)$. From elementary algebraic topology, given loops $a, b \in \Omega X$ we have a product $- \cdot - : \Omega X \times \Omega X \to \Omega X$ defined intuitively by traversing a, then b. Formally $a \cdot b$ is the projection under $I = (0,1) \to S^1$ of the map $I \to X$ defined by

$$(a \cdot b)(t) = \begin{cases} a(2t) & t < \frac{1}{2} \\ b(2t-1) & t \geq \frac{1}{2}. \end{cases}$$

However, this operation is not associative; given $a, b, c \in \Omega X$ certainly $(a \cdot b) \cdot c \neq a \cdot (b \cdot c)$ generally: diagrammatically

$(a \cdot b) \cdot c =$  $= a \cdot (b \cdot c).$

The classical resolution to this "problem" is to descend this multiplication to one on the homotopy classes of based maps—for which it is easy to see that associativity holds. We have of course re-discovered the group operation of the fundamental group $\pi_1(X)$ of X. Nonetheless before taking homotopy classes the product is *coherently associative*; there is a canonical homotopy turning a product associated one way into a product associated a different way, and canonical higher homotopies between these homotopies, and so on. The purpose of the little k-cubes operads is to recognise the fact that it can be undesirable to lose knowledge of this "coherence" structure if it is not necessary.

Definition 4.1. Define a topological category ${}^t\mathbb{E}_k^\otimes$ for each $k \geq 0$, called (by a slight abuse) the *little k-cubes operad*, with

- objects the objects $[n]$ of Fin_*, and
- a morphism $\alpha : [n] \to [m]$ in ${}^t\mathbb{E}_k^\otimes$ the data of a morphism $\hat{\alpha} \in \mathsf{Fin}_*([n] \to [m])$, along with a family of *rectilinear embeddings* (setting $\square = (0,1)$)

  $$\alpha_j : \square^k \times \alpha^{-1}\{j\} \to \square^k$$

  indexed by $j \in \underline{m}$.

  A rectilinear embedding $\square^k \times S \to \square^k$ for S a finite set is just a topological embedding (equipping S with the discrete topology) for which the projection onto each of the k components of $\square^k$, for each fixed $s \in S$, is given by a (possibly different) map $x \mapsto a + bx$.

- The set $\mathrm{Rect}(\square^k \times S \to \square^k)$ (or just $\mathrm{Rect}_k(S, \bullet)$) of rectilinear maps $\square^k \times S \to \square^k$ becomes a topological space when equipped with the compact-open topology (intuitively this makes continuous sliding of the "image k-cubes" continuous). Given a morphism $\alpha : [n] \to [m]$ in ${}^t\mathbb{E}_k^\otimes$ we regard it as an element of the topological space
$$\mathrm{Rect}_k(\hat{\alpha}) = \prod_{j \in \underline{m}} \mathrm{Rect}_k(\hat{\alpha}^{-1}\{j\}, \bullet).$$
In this way the morphism set ${}^t\mathbb{E}_k^\otimes([n] \to [m])$ can be made into a topological space by taking the disjoint union of the spaces $\mathrm{Rect}_k(\hat{\alpha}, \bullet)$ over all $\hat{\alpha} \in \mathsf{Fin}_*([n] \to [m])$.

- We have actually just unravelled the definition of the category ${}^t\mathbb{E}_k^\otimes$ obtained from an honest coloured operad ${}^t\mathbb{E}_k$ with multiplication spaces $\mathrm{Mul}_{{}^t\mathbb{E}_k}([\bullet]_{i \in \underline{n}}, \bullet) = \mathrm{Rect}_k(\underline{n}, \bullet)$. Composition of morphisms in ${}^t\mathbb{E}_k^\otimes$ is accomplished by using the "insertion" operation of the underlying operad ${}^t\mathbb{E}_k$. Concretely, this means that $\alpha : [n] \to [m]$ and $\beta : [m] \to [l]$ are composed by taking $\hat{\beta} \circ \hat{\alpha} : [n] \to [l]$ and equipping it with the family of rectilinear embeddings (for $p \in \underline{l}$)
$$(\beta \circ \alpha)_k : \square^k \times (\widehat{\beta \circ \alpha})^{-1}\{p\} \to \square^k \times \coprod_{j \in \hat{\beta}^{-1}\{p\}} \alpha^{-1}\{j\} \to \square^k$$
constructed from those of α and β in the obvious way.

Since each ${}^t\mathbb{E}_k^\otimes$ is a topological category, its nerve $\mathbb{E}_k^\otimes$ is automatically (by Corollary 1.1.5.12 of [11]) an ∞-category. We immediately obtain a forgetful functor $p : \mathbb{E}_k^\otimes \to N\mathsf{Fin}_*$ by setting $p(\alpha) = \hat{\alpha}$ for every morphism α. In fact, as should by now be obvious by its name, this functor confers unto $\mathbb{E}_k^\otimes$ the structure of an ∞-operad.

Theorem 4.2 (5.1.0.3 of [10])**.** *The evident forgetful functor $p : \mathbb{E}_k^\otimes \to NFin_*$ is an ∞-operad.*

Proof. The proof is a direct argument built on top of general ∞-operad theory. It proceeds by using the fact that $\mathbb{E}_k^\otimes$ is given by the so called *operadic nerve* of an ordinary (1-categorical) simplicial coloured operad $\mathcal{O}$—a notion which we have avoided explicitly introducing. Nonetheless, the direct definition of $\mathbb{E}_k^\otimes$ given above is a simple unwrapping of this construction. The multiplication spaces of this simplicial coloured operad $\mathcal{O}$ are given by
$$\mathrm{Mul}_{\mathcal{O}}([\bullet]_{i \in \underline{n}}, \bullet) = \mathrm{Sing}\,\mathrm{Rect}(\underline{n}, \bullet),$$
i.e. have multiplication spaces obtained by applying the Sing functor to those of ${}^t\mathbb{E}_k$ (this makes a *simplicial* coloured operad from a topological one). We then stand on the shoulders of the theory and appeal to the following general proposition, from which the theorem follows as a special case. □

Proposition 4.3. *If $\mathcal{O}$ is a fibrant*[12] *simplicial coloured operad, then its operadic nerve $N^\otimes \mathcal{O}$ is an ∞-operad.*

Proof. The proof is by inspection, directly observing that each of the three conditions of Definition 3.1 hold. Point (1) follows from restriction along the given morphism, and the functors of points (2) and (3) are easily seen to actually be isomorphisms in this case. See Proposition 2.1.1.27 of [10] for the details. □

[12] A *fibrant* simplicial coloured operad is one with every multiplication simplicial set fibrant.

We now return to the specific case of $k = 1$. Observe that for each $j \in \underline{m}$ a morphism $\alpha : [n] \to [m] \in {}^t\mathbb{E}_1^{\otimes}([n] \to [m])$ gives rise to a linear order on the set $\hat{\alpha}^{-1}\{j\}$; namely, the component map $\alpha_j : \square \times \hat{\alpha}^{-1}\{j\} \to \square$ is an embedding for each j, and therefore "reading across" the images of each $\square \times \{i\}$ with $i \in \hat{\alpha}^{-1}\{j\}$ over the interval $\square$ from left-to-right provides the desired linear ordering. Modelling the set of linear orderings on $\hat{\alpha}^{-1}\{j\}$ as a finite subset of $\mathrm{Rect}_1(\hat{\alpha}^{-1}\{j\}, \bullet)$ it is even clear that $\mathrm{Rect}_1(\hat{\alpha}^{-1}\{j\}, \bullet)$ deformation retracts onto this discrete subset. By the definition of the multiplication sets of the operad **Assoc**, we thus have a homotopy equivalence of topological spaces

$$\mathrm{Rect}_1(\hat{\alpha}^{-1}\{j\}, \bullet) \simeq \mathrm{Mul}_{\mathbf{Assoc}}([\bullet]_{i \in \underline{|\hat{\alpha}^{-1}\{j\}|}}, \bullet).$$

Since by definition

$${}^t\mathbb{E}_1^{\otimes}([n] \to [m]) = \coprod_{\hat{\alpha}:[n]\to[m]} \prod_{j \in \underline{m}} \mathrm{Rect}_1(\hat{\alpha}^{-1}\{j\}, \bullet),$$

there is therefore a weak equivalence of the topological categories ${}^t\mathbb{E}_1^{\otimes}$ and $\mathbf{Assoc}^{\otimes}$ (the latter having discrete homsets). This immediately[13] gives rise to an equivalence of their homotopy coherent nerves (considering each as a topological category) $\mathbb{E}_1^{\otimes}$ and $\mathrm{Assoc}^{\otimes}$, which resolves the question of the relationship between $\mathbb{E}_1^{\otimes}$ and $\mathrm{Assoc}^{\otimes}$ algebras; they are equivalent[14].

Thus $\mathbb{E}_1^{\otimes}$ gives the theory of associative algebras. The ∞-operad $\mathbb{E}_k^{\otimes}$ should then be thought of as specifying algebras with k coherent associative multiplications, since (with the tensor product of ∞-operads suitably defined) it is a famous theorem[15] that $\mathbb{E}_k^{\otimes} \otimes \mathbb{E}_{k'}^{\otimes} = \mathbb{E}_{k+k'}^{\otimes}$. There is a map of operads $\mathbb{E}_k^{\otimes} \to \mathbb{E}_{k+1}^{\otimes}$ called *stabilisation* and induced for each $k \geq 0$ by crossing morphisms with $\square$, and by Proposition 5.1.1.4. of [10] the colimit of the infinite sequence $\mathbb{E}_1^{\otimes} \to \mathbb{E}_2^{\otimes} \to \cdots$ is $\mathrm{Comm}^{\otimes}$. This justifies setting $\mathbb{E}_{\infty}^{\otimes} = \mathrm{Comm}^{\otimes}$.

The (1-categorical) topological operads ${}^t\mathbb{E}_k$ encode some fundamental structure in algebraic topology. One manifestation of this is the straightforward operadic action of ${}^t\mathbb{E}_k$; let $a_i \in \Omega^k X$ for $1 \leq i \leq n$, and let $c \in \Omega^k X$ be the constant k-loop which remains at the distinguished point of X. More precisely, each n-ary operation α of ${}^t\mathbb{E}_k$ which lies over the map $\hat{\alpha} : [n] \to [1]$ defined by $\hat{\alpha}(i) = 1$ for all $i > 0$ then gives a recipe for how to construct a new loop $f(a_1, \ldots, a_n) \in \Omega^k X$. Such an operation directly corresponds to a map $f \in \mathrm{Rect}(\square^k \times \underline{n} \to \square^k)$, and we construct $f(a_1, \ldots, a_n) \in \Omega^k X$ by deleting $\mathrm{im} f$ from the domain of c and then inserting a_i into the $\square^k$-hole left when we specifically deleted $f(\square^k \times \{i\})$. The result is a continuous k-loop, since each of the a_i are k-loops themselves.

We conclude this section by highlighting a classical theorem of May[13] which shows that not only does $\Omega^k X$ carry an ${}^t\mathbb{E}_k$-operadic action, but in fact this *characterises* the notion of the kth iterated loop space up to weak homotopy equivalence. It illustrates the fundamental role which the operads ${}^t\mathbb{E}_k$ play in algebraic topology.

Theorem 4.4 (May Recognition Theorem)**.** *Suppose that $X \in Top_*$ carries an operadic action of ${}^t\mathbb{E}_k$ and is connected. Then X is weakly homotopy equivalent to the kth iterated loop space of some pointed topological space.*

A direct analog of Theorem 4.4 theorem holds in the ∞-setting[16], and more-

[13] By 5.1.0.7 of [10].

[14] This is in contrast to the comparison between notions of associativity built from $N\mathbf{\Delta}^{\mathrm{op}}$ and $\mathrm{Assoc}^{\otimes}$, where each construction merely gave rise to the same algebra objects.

[15] A version of which is proved in [9].

[16] In [9] Lurie provides a generalisation of May's result to k-tuply monoidal ∞-stacks.

over the ∞-operads $\mathbb{E}_k^\otimes$ are of fundamental importance across many areas of mathematics; for instance $\mathbb{E}_\infty^\otimes$-algebra objects in the symmetric monoidal ∞-category of abelian groups recover the commutative rings, while the $\mathbb{E}_1^\otimes$-algebras are the non-commutative rings. Extending Theorem 4.2, it is additionally true that the ∞-operads $\mathbb{E}_k^\otimes$ are *coherent* (this is Theorem 5.1.1.1 of [10]), and therefore possess a rich theory of modules (and therefore support commutative algebra), as well.

5 The punchline for stable homotopy theory

Returning to our point of original inspiration, we are now ready to make our final definition and answer the question of the introduction.

Definition 5.1. An $\mathbb{E}_k$-*ring* (sometimes, a $\mathbb{E}_k$-*ring spectrum*) is an $\mathbb{E}_k^\otimes$-algebra object of Sp, the stable ∞-category of spectra. Thus, $\mathbb{E}_k$-rings are central objects of great interest in stable homotopy theory.

The ∞-category Sp is symmetric monoidal under the smash product of spectra, and the product is characterised by requiring that it preserves colimits in each variable and has tensor unit the sphere spectrum.

Since the $\mathbb{E}_k^\otimes$ operads are coherent, the general theory provides access to a robust formulation of commutative algebra applicable to the setting of spectra. For example, one has access to localisations of modules, and even a generalised localisation sending $\mathbb{E}_k^\otimes$-modules to $\mathbb{E}_{k+1}^\otimes$-modules! Tensor products of bimodules can be constructed, and spectral sequences can be extracted. Regarding an ordinary ring R as an $\mathbb{E}_1$-algebra, its derived category can be recovered by taking the homotopy category of its R-module spectra; this all generalises commutative and homological algebra (as in [10]).

The theory expands out in every direction. There is the so-called ∞-*operadic model structure* on the combinatorial simplicial ∞-category of "preoperads", of which the ∞-operads are the underling ∞-category. On the other hand, in the specific case of Sp, Goodwillie's "calculus of functors" [4] applies to functors $f : \mathsf{Sp} \to \mathsf{Sp}$ which preserve filtered colimits, allowing them to be approximated arbitrarily well by functors which are "Taylor polynomials" in an extremely concrete sense. The incredibly close resemblance paid by all of these generalised constructions to their classical counterparts is striking.

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