Transfers and tom Dieck splitting

via the Wirthmüller isomorphism

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In this report we give an essentially self-contained tour through the tom Dieck splitting theorem and its critical ingredient the Wirthmüller isomorphism, given some background in the basic theory of equivariant orthogonal spectra. We concentrate on the case of finite groups, since this considerably simplifies the exposition while key ideas are preserved. We highlight the category-theoretic formalism into which our Wirthmüller isomorphism fits as a special case, while also actually spelling-out the details and argument at the level of the basic maps used to build the construction. tom Dieck's original induction over conjugacy classes of subgroups is then applied to deduce the splitting theorem at the level of homotopy groups by reduction to a simple case, where all but one of the summands in the statement of the theorem are zero.

The tom Dieck splitting theorem and the Wirthmüller isomorphism are key results on the structure of the category of G-spectra and its homotopy category, and have several generalisations in the context of equivariant stable homotopy theory and more broadly. We could for example pass to the compact Lie group case, generalise tom Dieck splitting to functors other than that which takes homotopy fixed points, or could instead consider the Wirthmüller isomorphism at a higher level of abstraction—as a category-theoretic consequence of a particular context of adjoint functors and compatibility data.

1 Basic constructions

We begin by introducing our basic notation and some basic properties of the model of equivariant spectra with which we shall be working. Throughout we fix a finite group[1] G, with an ambient subgroup $H \le G$ often present. All of our containments ($\subset$ and $\subseteq$) will be strict, and we use e to denote the trivial subgroup of G. We let $\mathrm{N}H$ denote the normaliser of H in G, and let $\mathrm{W}H$ denote the associated *Weyl group* $\mathrm{W}H = \mathrm{N}H/H$. We use $\mathscr{C}_G$ to denote the set of conjugacy classes of subgroups of a group G.

Given a G-equivariant orthogonal spectrum X (which we will usually just call a G-spectrum), to each G-representation V is associated a G-space $X(V)$, which we sometimes call the *V-piece* of X (we will be working a complete G-universe). All of our G-spaces will be pointed and live in the category $G\mathsf{Top}_*$.

Definition 1.1. Given a G-space X and a point $x \in X$, define the *isotropy subgroup* G_x of G by

$$G_x = \{g \in G : g \cdot x = x\}.$$

Given the universal G-space EG for a finite group G, we can form a cofiber sequence

$$EG_+ \longrightarrow S^0 \longrightarrow \widetilde{EG}$$

called the *isotropy separation sequence* for G. It will be of great utility for us to generalise this construction in the sequel.

Definition 1.2. Let S be a set of subgroups of a group G which is closed under taking subgroups and under conjugation, and denote the set of all such S by $\mathscr{F}_G$. Then the *universal space* ES for S is a G-space characterised up to G-equivariant homotopy equivalence by the property that for all $H \le G$ the space $(ES)^H$ is empty if $H \notin S$ and is contractible if $H \in S$. It is a direct consequence of the universal property of such spaces that a map $\iota : ET \to ES$ is unique up to G-homotopy whenever[2] $T \subseteq S$.

[1] A comprehensive account in the generality of the compact Lie group case can be found for example in [11].

[2] In fact, this holds whenever there is a map $X \to ES$ with X having S-isotropy.

When S is instead a set of conjugacy classes of subgroups of G which is closed under taking subgroups (in the sense that if $[H] \in S$ and $K \leq H$ then we always have $[K] \in S$), we interpret S in ES as the set of all underlying subgroups. Of course, we should verify that we can construct ES in each case;

Lemma 1.3. *When S is a set of subgroups of a group G closed under taking subgroups and conjugation, the space ES actually exists.*

Proof. An elegant though abstract proof is obtained by direct appeal to Elmendorf's theorem (i.e that there is a Quillen equivalence between the categories of pointed G-spaces and ordinary space-valued presheaves on the orbit category $\mathcal{O}^G$). Indeed, given a subset $S \subseteq \mathscr{C}_G$ we can define a presheaf F on $\mathcal{O}^G$ by sending G/H to the empty set if $H \notin S$ and the one-point set if $H \in S$, and this is a functor exactly because of the conditions we have placed on S. Elmendorf's theorem then immediately supplies ES as the G-space corresponding to F. □

This is not to say that the space ES cannot posses a very concrete description. For example, if we fix a representation V of G then [12] provides a construction of ET_V where T_V is the set of subgroups H of G which have V^H nonzero (necessarily $T_V \in \mathscr{F}_G$); let S^n_V denote the unit sphere in $\bigoplus_{i=1}^n V$ (not the one-point compactification). Now consider the G-space $S^\infty_V = \bigcup_{n=1}^\infty S^n_V$ obtained via the inclusion $S^n_V \hookrightarrow S^{n+1}_V$ induced by the inclusion of the first n summands of V into an $(n+1)$-fold direct sum of V with itself. When $H \leq G$ is such that $H \in T_V$ then $(S^\infty_V)^H$ is just the infinite-dimensional sphere $S^\infty_{V^H}$ and hence is contractible, and when $H \notin T_V$ the same formula for $(S^\infty_V)^H$ shows that it is empty. Therefore S^∞_V is the desired model for ET_V.

2 The category of orthogonal G-spectra

In this section we recall and assimilate useful properties of the category of orthogonal G-spectra and its functors.

The category Sp^G of orthogonal G-spectra becomes symmetric monoidal when it is equipped with the *smash product* $- \wedge -$ of spectra. In its typical incarnation, the smash product[3] of equivariant orthogonal spectra is just the smash product of the underlying orthogonal spectra equipped with the diagonal action. The non-equivariant product can then be obtained as the Day convolution product for functors on monoidal categories of orthogonal groups which are enriched over topological spaces (this is covered in detail in [9]).

For each G-spectrum X and G-representation V recall that we have a suspension functor Σ^V and a loop functor Ω^V, and for Y a G-space we can form the suspension spectrum $\Sigma^\infty Y \in \mathsf{Sp}^G$. The functors Σ^V and Ω^V arise since Sp^G is tensored and cotensored over $G\mathsf{Top}_*$ (the category of pointed G-spaces), and hence we can take smash products $\Sigma^V X = S^V \wedge X$ and mapping spaces $\mathrm{Map}^G(S^V, X)$ between G-spectra and honest G-spaces. In each of these constructions there is an evident homeomorphism from the respective U-pieces of $S^V \wedge X$ and $\mathrm{Map}^G(S^V, X)$ (with U just some G-representation), and the G-spaces $S^V \wedge X(V)$ and $\mathrm{Map}^G(S^V, X(V))$, and we freely pass between these usually without explicit comment (a detailed account of the construction of this homeomorphism can be found in the appendix of [4]).

The nth homotopy group of a G-spectrum X is defined by the colimit formulas

$$\pi_n^H(X) = \begin{cases} \mathrm{colim}_V\, \pi_n^H(\Omega^V X(V)) & n \geq 0 \\ \mathrm{colim}_V\, \pi_0^H(\Omega^{V-\mathbf{R}^{-n}} X(V)) & \text{otherwise} \end{cases} = \begin{cases} \mathrm{colim}_V [S^V \wedge S^n, X(V)]^H & n \geq 0 \\ \mathrm{colim}_V [S^{V-\mathbf{R}^{-n}}, X(V)]^H & \text{otherwise} \end{cases}.$$

While an entire universe of G-representations can be rather unwieldy, the following lemma greatly eases our burden when studying families of maps which assemble to maps of homotopy groups.

Lemma 2.1. *Let R be the regular representation of G. Then the natural infinite sequence of inclusions $R \to R \oplus R \to R \oplus R \oplus R \to \ldots$ is cofinal in the filtered diagram of the complete G-universe.*

We will often combine the following two lemmas in order to reason about pullbacks obtained from the pushouts used to build G-CW-complexes.

Lemma 2.2. *There is a model structure on G-spectra where the weak equivalences $f : X \to Y$ are the morphisms which induce isomorphisms of all homotopy groups, and this condition is equivalent to asking that the map of fixed points $f^H : X^H \to Y^H$ is a weak equivalence for every $H \leq G$.*

[3] Passing to the homotopy category, this product is uniquely characterised in [14] up to natural isomorphism (assuming mild conditions) by requiring that it preserves colimits in each variable and has tensor unit the equivariant sphere spectrum.

Lemma 2.3. *This model category structure on Sp^G is right proper, meaning that in every pullback square*

$$\begin{array}{ccc} X & \dashrightarrow & Y \\ \vdots & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

the pullback of a weak equivalence along a fibration is[4] *a weak equivalence.*

Another very useful (although standard) property of the mapping space functor is given below.

Proposition 2.4. *For $X \in GTop_*$, the functor $\mathrm{Map}^G(-,X)$ sends colimits to limits.*

Finally, we recall the natural forgetful functor from G-spectra to H-spectra and its adjoints.

Proposition 2.5. *Let $H \le G$ be a subgroup. There is a restriction functor*

$$U_H : Sp^G \to Sp^H,$$

which has a left-adjoint functor (induction)

$$G \wedge_H - : Sp^H \to Sp^G,$$

and a right-adjoint functor (coinduction)

$$\mathrm{Map}^H(G,-) : Sp^H \to Sp^G.$$

Note that one might worry that this is not well defined, or at least that the functor U_H takes G-spectra over the complete G-universe into H-spectra over the universe of H-representations which are restrictions of G-representations. Formally this is dealt with via a point-set change-of-universe functor (which is considered in detail in [8]), but the subtleties which arise when constructing such a functor and then when applying it will not be relevant for our[5] purposes.

Finally, recall that for V any G-representation and $H \le G$ there is a non-canonical space level transfer map $\tau_H^G : S^V \to G \wedge_H S^V$ obtained by embedding G into V so that cosets of H go to disjoint open unit balls of V.

3 External transfer via the Wirthmüller isomorphism

One interpretation of the Wirthmüller isomorphism ([1]) is that it asserts the equivariant version of the isomorphism

$$\bigvee_{i=1}^n X_i \xrightarrow{\sim} \prod_{i=1}^n X_i$$

in the homotopy category of ordinary spectra. It will play a critical role in the program for proving tom Dieck splitting, via the *external transfer map* $T_H^G : \pi_\bullet^H(X) \to \pi_\bullet^G(G \wedge_H X)$ which it will permit us to define for each H-spectrum X. We will also be able to relate T_H^G to the so-called *internal transfer maps* $t_H^G : \pi_\bullet^H(X) \to \pi_\bullet^G(X)$ when X is a G-spectrum, connecting T_H^G to the geometric Thom–Pontryagin construction.

This brings us to the statement of the theorem.

Theorem 3.1 (Wirthmüller isomorphism)**.** *For every $H \le G$ and H-spectrum X there is a map*

$$\Phi : G \wedge_H X \to \mathrm{Map}^H(G,X)$$

inducing an isomorphism of homotopy groups.

The actual isomorphism which we will construct has a categorical interpretation; under suitable hypotheses, the existence of the isomorphism itself follows by essentially formal reasons; the hard work, then, is spent showing that the hypotheses are actually satisfied in the equivariant stable category. While we will not explicitly adopt that approach in order to establish the existence of the isomorphism here, the statement of the category-theoretic result serves to clarify the situation and emphasise the important pieces, so we at least introduce their definitions now.

[4] This follows directly from the fact that weak equivalences are sent to isomorphisms by π_n, and π_n preserves limits (hence pullbacks).

[5] For us, it will be enough to be observe that the restriction of the regular representation of G to H gives $[G : H]$ copies of the regular representation of H, and then appeal to Lemma 2.1.

From the perspective of the equivariant stable category, our motivation is the following. Recall as we saw above that for each subgroup $H \le G$ we have a restriction functor $U_H : \mathsf{Sp}^G \to \mathsf{Sp}^H$, and in addition this functor is flanked on either side by adjoint functors[6]; we have a left-adjoint $G \wedge_H - : \mathsf{Sp}^H \to \mathsf{Sp}^G$, and a right-adjoint $\mathrm{Map}^H(G,-) : \mathsf{Sp}^H \to \mathsf{Sp}^G$.

Now, the category Sp^K for any $K \le G$ is symmetric monoidal when equipped with the smash product of spectra, and is a closed category since it has the internal-hom functor $\mathrm{Map}^K(-,-)$. In fact, as the following definition–theorem pair explains, the Wirthmüller isomorphism already almost follows formally;

Definition 3.2. Let $(\mathscr{A}, \otimes_{\mathscr{A}}, \mathbf{1}_{\mathscr{A}})$ and $(\mathscr{B}, \otimes_{\mathscr{B}}, \mathbf{1}_{\mathscr{B}})$ be closed symmetric monoidal categories, and suppose that there are functors $F : \mathscr{A} \to \mathscr{B}$ and $F_!, F_* : \mathscr{B} \to \mathscr{A}$ with $F_! \dashv F$ and $F \dashv F_*$.

When F is a monoidal functor (that is, preserves tensor products and the tensor unit up to natural isomorphism[7], satisfying the necessary compatibility diagrams) the triple $(F_!, F, F_*)$ is called a *pre-Wirthmüller context.*

If in addition F is closed (that is, F also preserves internal-homs up to natural isomorphism with the necessary diagrams commuting), then the triple $(F_!, F, F_*)$ is promoted to a *Wirthmüller context.*

Theorem 3.3 (Generalised Wirthmüller isomorphism)**.** *Let $(F_!, F : \mathscr{A} \to \mathscr{B}, F_*)$ be a Wirthmüller context. Then if $D \in \mathscr{B}$ is such that $F_! \mathbf{1}_{\mathscr{B}}$ has dual $F_! D$, then there exists a natural isomorphism of functors*

$$F_* F \longrightarrow F_!(F \otimes D),$$

where $F \otimes D$ denotes the composition $(- \otimes D) \circ F$.

When $(F_!, F, F_*) = (G \wedge_H -, U_H, \mathrm{Map}^H(G,-))$ then $F_! \mathbf{1}_{\mathscr{B}}$ is self-dual and for each $X \in \mathsf{Sp}^G$ we recover the desired isomorphism

$$\mathrm{Map}^H(G, X) \xrightarrow{\;\sim\;} G \wedge_H (X \wedge \mathbb{S}) \cong G \wedge_H X$$

(here we have implicitly applied U_H to X where it appears).

This situation is closely related to the analogous situation in the representation theory of finite groups; in the case of finite-dimensional representations F is a again the restriction functor and the functors $F_!$ and F_* both happen to be isomorphic (and given by the induction functor). One can view this as a consequence of the fact that finite direct sums and finite direct products here are themselves isomorphic.

In the case of G-spectra the situation is not so straightforward. In fact, the proof found in the standard (although slightly dated) text [4] is incorrect and requires a nontrivial modification in order to repair[8]. A proof which proceeds via these abstract methods appears in [10]. Blumberg [1] gives a more intuitive proof which holds in the special case of a connected or bounded-below spectrum. We adopt the hands-on approach of [12], directly constructing the map from simple pieces.

At the core of the isomorphism, we directly build a G-map $\Phi : G \wedge_H X \to \mathrm{Map}^H(G, X)$ for X a pointed H-space by the formula (for $g' \in G$)

$$\Phi(g,x)(g') = \begin{cases} g'g \cdot x & g'g \in H \\ x_0 & \text{otherwise} \end{cases}$$

with x_0 the basepoint of X. The map Φ is G-equivariant since for $k \in G$ we have

$$\Phi(k \cdot g, x)(g') = \begin{cases} g'kg \cdot x & g'kg \in H \\ x_0 & \text{otherwise} \end{cases} = \Phi(g,x)(g'k) = (k \cdot \Phi(g,x))(g').$$

There is another pair of named maps which will together prove very useful to clarify the situation; the first is the so-called *assembly map*, which encodes is the natural way to commute the covariant map functor with smashes. Namely, for X and Y both pointed G- and H-spaces respectively, elements $(f, y) \in X \wedge \mathrm{Map}^H(G, Y)$ can be sent to elements of $\mathrm{Map}^H(G, X \wedge Y)$ by

$$\alpha(x, f) : g \mapsto (gx, f(g)),$$

where we note that in the smash $X \wedge Y$ we implicitly apply the forgetful functor from G-spaces to H-spaces to X.

The second interesting map is the isomorphism of G-spaces $X \wedge (G \wedge_H Y) \to G \wedge_H (X \wedge Y)$ defined by (again X is implicitly regarded as an H-space)

$$\sigma : (x, (g, y)) \mapsto (g, (g^{-1}x, y)),$$

[6] At the level of 2-categories this is the data of an adjunction of adjunctions, but we do not dwell on this any further.

[7] It is sometimes the convention in algebraic topology and elsewhere to insert the adjective "strong" to indicate that these compatibility maps must be isomorphisms, but we stick to the category-theorist's convention of implying that meaning by default.

[8] See [1].

called the *shearing map*. This map is clearly G-equivariant, and the fact that it is an isomorphism is a direct corollary of the calculation in $G \wedge_H (X \wedge Y)$ that if $g' = gh$ for $g, g' \in G$ and $h \in H$ then $(g', (g'^{-1}x, y)) = (gh, (h^{-1}g^{-1}x, y) = (g, (hh^{-1}g^{-1}x, hy) = (g, (g^{-1}x, hy))$.

By combining these two maps with Φ above, together they provide two different ways to obtain maps $X \wedge (G \wedge_H Y) \to \mathrm{Map}^H(G, X \wedge Y)$, namely via the composites

$$X \wedge (G \wedge_H Y) \xrightarrow{\mathrm{id}_X \wedge \Phi} X \wedge \mathrm{Map}^H(G, Y) \xrightarrow{\alpha} \mathrm{Map}^H(G, X \wedge Y)$$

and

$$X \wedge (G \wedge_H Y) \xrightarrow{\sigma} G \wedge_H (X \wedge Y) \xrightarrow{\Phi} \mathrm{Map}^H(G, X \wedge Y).$$

In fact, we can just directly compute that for any $g' \in G$ (and $z_0 \in X \wedge Y$ the basepoint) we have

$$\begin{aligned}(\alpha \circ (\mathrm{id}_X \wedge \Phi))(x, (g, y))(g') = \alpha(x, \Phi(g, y))(g') = (g'x, \Phi(g, y)(g')) &= \begin{cases} (g' \cdot x, g'g \cdot y) & g'g \in H \\ (g' \cdot x, y_0) & \text{otherwise} \end{cases} \\ &= \begin{cases} (g' \cdot x, g'g \cdot y) & g'g \in H \\ z_0 & \text{otherwise} \end{cases}\end{aligned}$$

and similarly

$$\begin{aligned}(\Phi \circ \sigma)(x, (g, y))(g') = \Phi(g, (g^{-1}x, y))(g') = \begin{cases} g'g \cdot (g^{-1}x, y) & g'g \in H \\ z_0 & \text{otherwise} \end{cases} &= \begin{cases} (g'gg^{-1} \cdot x, g'g \cdot y) & g'g \in H \\ z_0 & \text{otherwise} \end{cases} \\ &= \begin{cases} (g' \cdot x, g'g \cdot y) & g'g \in H \\ z_0 & \text{otherwise} \end{cases},\end{aligned}$$

which exactly says that these composites give the same map.

There is one more map (really, composite of maps) which we need to consider, and which has a slightly more complicated origin; fix a G-representation V. Then we saw above that there is a transfer map $\tau_H^G : S^V \to S^V \wedge_H G$ (once again we everywhere omit applications of the forgetful functor from G-spaces to H-spaces). We can use it to build the *cross map* δ out of the composite (here ε_Y is the adjunction between U_H and $\mathrm{Map}^H(G, -)$, giving a map $\mathrm{Map}^H(G, Y) \to Y$)

$$\begin{aligned}\delta : S^V \wedge \mathrm{Map}^H(G, Y) \xrightarrow{\tau_H^G \wedge \mathrm{id}_{\mathrm{Map}^H(G,Y)}} (G \wedge_H S^V) \wedge \mathrm{Map}^H(G, Y) &\xrightarrow{\quad\sigma\quad} G \wedge_H (S^V \wedge \mathrm{Map}^H(G, Y)) \\ &\xrightarrow{G \wedge_H (\mathrm{id}_{S^V} \wedge \varepsilon_Y)} G \wedge_H (S^V \wedge Y).\end{aligned}$$

Of central importance is the fact that all of the maps we have seen assemble together to give composites which respect one-another up to G-homotopy, as codified in the following lemma.

Lemma 3.4. *Specialising to the case of $X = S^V$ above, we have a diagram*

$$\begin{array}{ccc} S^V \wedge (G \wedge_H Y) & \xrightarrow{\mathrm{id}_{S^V} \wedge \Phi} & S^V \wedge \mathrm{Map}^H(G, Y) \\ \downarrow{\scriptstyle \sigma} & \swarrow{\scriptstyle \delta} & \downarrow{\scriptstyle \alpha} \\ G \wedge_H (S^V \wedge Y) & \xrightarrow{\Phi} & \mathrm{Map}^H(G, S^V \wedge Y) \end{array}$$

where the outer square commutes strictly, and each triangle commutes modulo G-homotopy.

Proof. We directly checked strict commutativity of the outer square above. Commutativity of the triangles modulo G-homotopy is a direct calculation where in each case one just expands the definition of the transfer map and the respective left and right adjoints of the forgetful functor from G- to H-spaces permits the writing down of an explicit G-homotopy. Since we have not given an explicit construction of the maps τ_H^G, we neglect to do this here, and instead refer the reader who desires an explicit implementation of these details to Proposition 4.5 of [12]. □

With Lemma 3.4 now in-hand, the Wirthmüller isomorphism is within our reach. The proof uses repeated stabilisation as a crutch allowing it to proceed, and this illustrates why the Wirthmüller isomorphism does not hold at the space level.

Proof of Theorem 3.1. Let X be a H-spectrum for $H \leq G$. We will show that the map Φ defined above induces an isomorphism Φ_* on the homotopy groups of the spectra $G \wedge_H X$ and $\mathrm{Map}^H(G, X)$. Of course, as a map Φ really depends on a particular G-space, so induces a map of G-spectra which we will call by the same name. In order to mitigate a mess of subscripts which do not help to clarify the situation in the following diagrams, we use Φ interchangeably to refer to the family of maps obtained by varying X (all compositions with Φ will be completely unambiguous, with domains and codomains labelled on the diagrams to appear).

To this end observe that for each $k \geq 0$, $K \leq G$ any subgroup, and K-representation V the map $\Phi : G \wedge_H X \to \mathrm{Map}^H(G, X)$ induces a map

$$\Phi_{V*} : [S^V \wedge S^n, (G \wedge_H X)(V)]^K \to [S^V \wedge S^n, \mathrm{Map}^H(G, X)(V)]^K,$$

which as a family parameterised by V assemble to encode a map $\pi_k^K(G \wedge_H X) \to \pi_k^K(\mathrm{Map}^H(G, X))$.

We need to show that this map is both injective and surjective; each individual map Φ_{V*} need not be, but this must be case up to stabilisation. To see injectivity, let $\psi : S^V \wedge S^n \to (G \wedge_H X)(V)$ represent a class in $[S^V \wedge S^n, (G \wedge_H X)(V)]^K$ in the kernel of Φ_{V*}.

The main step is to view $(G \wedge_H X)(V)$ as $G \wedge_H X(V)$ and post-compose ψ with Φ_{V*}. However, this only makes sense if the K-representation V can be upgraded into a G-representation, and is a key reason why the Wirthmüller isomorphism does not hold at the space level. This need not actually be possible, but we can instead exploit cofinality of the sequence of iterated direct sums of the regular representation to assume that V is some number k of copies of the regular representation of K. Then we can replace V with k copies of the regular representation of G, whence V restricts to $k[G : K]$ copies of the regular representation of K.

The result of this process is a map $\Phi \circ \psi : S^V \wedge S^n \to \mathrm{Map}^H(G, X(V))$. Since ψ represents a class in the kernel of Φ_*, we can replace V again so that $\Phi \circ \psi$ is actually nullhomotopic through K-equivariant maps (care must be taken throughout to ensure that our composites are equivariant with respect to the correct group—in this case K). Taking the smash product with S^U for U any G-representation and then post-composing with the cross map δ gives a nullhomotopic composite

$$\begin{array}{ccccc} S^U \wedge S^V \wedge S^n & \xrightarrow{\mathrm{id}_{S^U} \wedge \psi} & S^U \wedge (G \wedge_H X(V)) & \xrightarrow{\mathrm{id}_{S^U} \wedge \Phi} & S^U \wedge \mathrm{Map}^H(G, X(V)) \\ & & \downarrow{\scriptstyle \sigma} & \swarrow{\scriptstyle \delta} & \\ & & G \wedge_H (S^U \wedge X(V)) & & \end{array},$$

with the dotted map added for our immanent convenience. The G-homotopy provided by Lemma 3.4 gives a homotopy from the entire composite to the map $\sigma \circ (\mathrm{id}_{S^U} \wedge \psi)$, and hence shows that this latter map is nullhomotopic as well. But σ is a homeomorphism, so we conclude that the original map ψ is nullhomotopic (perhaps after stabilisation), as desired. The case for $k < 0$ is essentially a duplication of this argument, so we omit it, and the situation is the same in the next part.

It remains to show surjectivity, and hence fix $\phi : S^V \wedge S^n \to \mathrm{Map}^H(G, X)(V)$ representing a class in $[S^V \wedge S^n, \mathrm{Map}^H(G, X)(V)]^K$ with V a K-representation. Exactly as before, by replacing V we can assume that it is actually a G-representation and so again smashing with S^U for U a G-representation and post-composing with the cross map δ and then Φ we get another diagram (the dotted maps will be shortly required as well, and here μ the corresponding structure map for the spectrum X)

$$\begin{array}{ccc} & & S^U \wedge S^V \wedge S^n \\ & & \downarrow{\scriptstyle \mathrm{id}_{S^U} \wedge \phi} \\ & & S^U \wedge \mathrm{Map}^H(G, X(V)) \\ & \swarrow{\scriptstyle \delta} & \downarrow{\scriptstyle \alpha} \\ G \wedge_H (S^U \wedge X(V)) & \overset{\Phi}{\dashrightarrow} & \mathrm{Map}^H(G, S^U \wedge X(V)) \\ \downarrow{\scriptstyle G \wedge_H \mu} & & \downarrow{\scriptstyle \mathrm{Map}^H(G, \mu)} \\ G \wedge_H X(U \oplus V) & \xrightarrow{\Phi} & \mathrm{Map}^H(G, X(U \oplus V)) \end{array}.$$

Here the bottom square arises just as a naturality square for the map Φ and commutes strictly, and $G \wedge_H \mu$ and $\mathrm{Map}^H(G, \mu)$ each just denote application of their respective functors to the structure map μ. Appealing to

Lemma 3.4 we immediately find that the solid composite along the left side of the diagram

$$\Phi \circ (G \wedge_H \mu) \circ \delta \circ (\mathrm{id}_{S^U} \wedge \phi)$$

is G-homotopic to the composite

$$\mathrm{Map}^H(G, \mu) \circ \alpha \circ (\mathrm{id}_{S^U} \wedge \phi),$$

since the triangle commutes up to G-homotopy. But the composite $\mathrm{Map}^H(G,\mu) \circ \alpha$ is just a structure map for the G-spectrum $\mathrm{Map}^H(G, X)$ by definition, so the commutativity of the entire diagram up to G-homotopy witnesses the fact that the homotopy class of the composite (the left vertical arm)

$$(G \wedge_H \mu) \circ \delta \circ (\mathrm{id}_{S^U} \wedge \phi)$$

is sent by Φ_{V*} (just post-composition with Φ, completing our path through the diagram) to the class of ϕ. Hence we conclude that the Wirthmüller map induced by Φ is also surjective, and this completes the proof. □

As a consequence, we get an isomorphism $\Phi_* : \pi_\bullet^K(G \wedge_H X) \to \pi_\bullet^K(\mathrm{Map}^H(G, X))$ for every $K \leq G$ and H-spectrum X. Specialising to the case $K = G$, we also have a H-equivariant map $\varepsilon_X : \mathrm{Map}^H(G, X) \to X$ given by the counit of the adjunction between the forgetful functor from G-spectra to H-spectra and $\mathrm{Map}^H(G, -)$, so at the level of homotopy groups we can form the composite of solid morphisms given by the diagram (the map R_H^G arises just because G-equivariant homotopies are also H-equivariant)

$$\begin{array}{ccccc}
\pi_\bullet^G(G \wedge_H X) & \xrightarrow{R_H^G} & \pi_\bullet^H(G \wedge_H X) & & \\
 & & \downarrow{\scriptstyle \Phi_*} & \searrow{\scriptstyle \rho_* = \varepsilon_{X*} \circ \Phi_*} & \\
 & & \pi_\bullet^H(\mathrm{Map}^H(G, X)) & \xrightarrow{\varepsilon_{X*}} & \pi_\bullet^H(X)
\end{array} \quad . \tag{1}$$

Since the counit map $\varepsilon_X : \mathrm{Map}^H(G, X) \to X$ is just evaluation at $e \in G$, by unravelling definitions we see directly that the dotted morphism is given induced on the V-piece by the explicit formula

$$\rho(g, x) = \begin{cases} g \cdot x & g \in H \\ x_0 & \text{otherwise} \end{cases}$$

for again x_0 the basepoint of X. Hence our original composite is given by a particularly simple formula. The content of the next proposition is that in fact this restriction-projection map sees everything.

Proposition 3.5 (Transfers)**.** *The composite* (1) *is an isomorphism. By taking its inverse, we obtain a canonical map*

$$T_H^G : \pi_\bullet^H(X) \to \pi_\bullet^G(G \wedge_H X).$$

Moreover, when X is actually a G-spectrum there is a natural map $m : G \wedge_H X \to X$ obtained by remembering the G-action and letting G act on X on the left, and post-composing with the map of homotopy groups induced by m gives a map

$$t_H^G : \pi_\bullet^H(X) \to \pi_\bullet^G(X).$$

Proof. The second part is immediate, so we just show the first. We just exploit the fact that a commuting square fits into (1); dropping the dotted composite, we have a commuting diagram

$$\begin{array}{ccccc}
\pi_\bullet^G(G \wedge_H X) & \xrightarrow{R_H^G} & \pi_\bullet^H(G \wedge_H X) & & \\
\downarrow{\scriptstyle \Phi_*} & & \downarrow{\scriptstyle \Phi_*} & & \\
\pi_\bullet^G(\mathrm{Map}^H(G, X)) & \xrightarrow{R_H^G} & \pi_\bullet^H(\mathrm{Map}^H(G, X)) & \xrightarrow{\varepsilon_{X*}} & \pi_\bullet^H(X)
\end{array} \quad .$$

We have just seen that the leftmost map is an isomorphism (it is the Wirthmüller isomorphism), and it can be seen that the composite along the bottom row is an isomorphism directly. Indeed, we just note that the adjunction giving rise to the counit ε_X asserts that there is an isomorphism of homotopy classes (here again we apply the relevant forgetful functors from G to H where necessary without explicit mention)

$$[S^V \wedge S^n, \mathrm{Map}^H(G, X(V))]^G \to [S^V \wedge S^n, X(V)]^H$$

for each $n \geq 0$ and G-representation V. Restriction of arbitrary G-representations V to H-representations does not hit every possible H-representation, but it does hit arbitrarily large direct sums of the regular representation of H, so cofinality permits passing to an isomorphism in the colimit. □

Definition 3.6. For X a H-spectrum, the canonical isomorphism $T_H^G : \pi_\bullet^H(X) \to \pi_\bullet^G(G \wedge_H X)$ provided in Proposition 3.5 by the Wirthmüller isomorphism is called the *external transfer map*, or sometimes just the *Wirthmüller isomorphism* itself.

When X instead a G-spectrum, the map $t_H^G : \pi_\bullet^H(X) \to \pi_\bullet^G(X)$ is called the *internal transfer map*.

The external transfer map can also be defined directly via the so-called Thom–Pontryagin construction [5]. The advantage of our approach is that many of the basic properties of the transfer can be seen to just follow from the description (1) in terms of a particularly simple composite.

4 The tom Dieck splitting theorem

In the category of G-spectra, a very natural operation is to take fixed points under a subgroup $H \leq G$, but there are a number of ways to perform this construction. We introduce the three fundamental notions of doing so below.

Definition 4.1. Let X be a G-spectrum. Then for $H \leq G$ we want (morally) to form an $\mathrm{N}H$-spectrum X^H by naively taking the H-fixed points on each V-piece of X (for V any G-representation). The right way to do this is by taking $\mathrm{Map}^G(G/H_+, X)$ in analogy with the same situation for pointed G-spaces, which defines a functor called taking the *categorical H-fixed points*.

In contrast, we could instead (morally) take for each V-piece $X(V)$ of X the G-space $\mathrm{Map}^H(EG_+, X(V))$ (observe that any EG is an EH), yielding the *homotopy fixed points* functor $\mathrm{Map}^H(EG_+, -)$. This is a natural construction since EG is analogous to the point G/G, in the sense that both spaces have contractible G-fixed points.

Finally, for $H \leq G$ there are the *geometric H-fixed points*, which are a bit more difficult to explicitly pin down. One definition (which requires an existence and uniqueness theorem) is that it is the unique left derived monoidal functor Φ^H which preserves homotopy colimits and satisfies $\Phi^H\Sigma^\infty X \cong \Sigma^\infty X^H$. (There are several other definitions, for example an explicit one defined by taking the smash product with the suspension spectrum of a canonical space, but we do not delve into that construction here.)

For $f : X \to Y$ a morphism of G-spectra, we let $f^H : X^H \to Y^H$ denote the induced map under the categorical H-fixed points functor. It is important to note that we must define (for example) categorical fixed points using our general mapping-space machinery, instead of directly assembling a new spectrum V-piecewise by taking the H-fixed points of honest G-spaces; we could still define a G-spectrum like this, but in general it would not behave nicely with respect to homotopy. For instance, if f is was weak equivalence then this other definition of the induced map f^H need not be a weak equivalence of non-equivariant spectra.[9]

In a similar vein, if X and Y are G-spectra and $H \leq G$ is any subgroup, it follows directly from the definition that taking the smash product and taking the H-fixed points commute, i.e.

$$(X \wedge Y)^H \cong X^H \wedge Y^H.$$

Since for each G-representation V the suspension spectrum $\Sigma^\infty X$ associates the G-space $S^V \wedge X$, one might hope that a similarly simple relation holds when commuting the suspension spectrum past taking categorical H-fixed points. Unfortunately, there is again no such relation; these operations do not commute, and the correct way to "distribute" the H-fixed points operator over a smash product is explained by the *tom Dieck splitting theorem* which we will shortly see.

At any rate, one begins to see the purpose of the alternative notions of fixed points which we have presented. For example, our definition of geometric H-fixed points manifestly commutes with taking the fixed points of a G-space in this way. We now pause to note the following basic result[10] on categorical fixed points, which we will apply often without further mention in the sequel. In fact, an identical result regarding the geometric fixed points functor also holds (see [4]).

Proposition 4.2. *Let $f : X \to Y$ be a map of G-spectra. Then f is a weak equivalence if and only if for each subgroup $H \leq G$ the map of categorical H-fixed points $f^H : X^H \to Y^H$ is a weak equivalence.*

The purpose of the remainder of this section is to use tom Dieck splitting to clarify the situation regarding the categorical fixed points of a suspension spectrum. At the level of spectra, it says

Theorem 4.3 (tom Dieck Splitting, spectrum version)**.** *For X a G-space, there is a map*

$$(\Sigma^\infty X)^G \longrightarrow \bigvee_{[H]\in\mathscr{C}_G} \Sigma^\infty(E\mathrm{W}H_+ \wedge_{\mathrm{W}H} X^H)$$

[9] This is explored in [12].

[10] e.g. in [1, 5].

inducing an isomorphism on all homotopy groups. Here "$\wedge_{\mathrm{W}H}$" denotes the quotient of the ordinary wedge by the W*H action.*

This version was first established in [4] in 1986. In the more modern account of [6], the theorem is instead deduced via the equivariant Barratt–Priddy–Quillen theorem as a purely category-theoretic consequence.

We will prove tom Dieck's original version [2] using the approach of [1, 12], which occurs at the level of stable homotopy groups:

Theorem 4.4 (tom Dieck Splitting, homotopy version)**.** *For X a G-space, we have*

$$\pi_\bullet^G(\Sigma^\infty X) \cong \bigoplus_{[H]\in\mathscr{C}_G} \pi_\bullet^{\mathrm{W}H}(\Sigma^\infty(E\mathrm{W}H_+ \wedge X^H)).$$

We begin with a definition which will feature prominently throughout the main argument.

Definition 4.5. Let X be a G-space and let $[H] \in \mathscr{C}_G$. Then X *is concentrated at* $[H]$ if whenever $K \le G$ is such that $K \not\in [H]$ then X^K is contractible.

In order to acquaint ourselves with the notion of being concentrated at a conjugacy class we now establish the following technical proposition, which will prove to be a critical piece of our proof of the main theorem. The proof proceeds by induction on the cells of a G-CW-complex, which contrasts with the other primary method of induction in equivariant stable homotopy theory which we will also shortly see—induction on the conjugacy classes $\mathscr{C}_G$ of G. The intervening lemma is clear;

Lemma 4.6. *Let $H \le G$ be a subgroup, and let $X, Y \in G\mathit{Top}_*$. Then* $\mathrm{Map}^G(G/H \wedge X, Y) \cong \mathrm{Map}^G(X, Y^H)$.

Proposition 4.7 (Warm-up)**.** *Let $H \le G$ be a normal subgroup and let $Y \in \mathit{Top}_*^G$ be concentrated at $[H] \in \mathscr{C}_G$. Then for every G-CW-complex X with finitely many cells the restriction map*

$$\rho : \mathrm{Map}^G(X, Y) \to \mathrm{Map}^{G/H}(X^H, Y^H)$$

is a weak equivalence and a fibration.

Proof. The map is defined by sending $f \in \mathrm{Map}^G(X, Y) : X \to Y$ to the induced map of H-fixed points $\tilde{f} : X^H \to Y^H$. We construct this map by induction on the cells of X; the strategy is to consider separately those cells $G/K \times S^k$ with $K \in [H]$, and those with $K \not\in [H]$. Thus we partition X into (non-disjoint) subcomplexes

$$X^H = \{x \in X : H \le G_x\} \quad \text{and} \quad X^{\not\le H} = \{x \in X : G_x \not\le H\},$$

noting that X^H is only a G-subcomplex because H is normal in G. The remainder of the G-CW-complex X can then be built from $X^H \cup X^{\not\le H}$, so necessarily the additional attached cells G/K have K properly contained in H. We can then construct ρ as the composite of the maps (all induced by the inclusions of spaces)

$$\mathrm{Map}^G(X, Y) \to \mathrm{Map}^G(X^H \cup X^{\not\le H}, Y) \to \mathrm{Map}^G(X^H, Y),$$

since we have the chain of equalities

$$\mathrm{Map}^G(X^H, Y) = \mathrm{Map}^G(X^H, Y^H) = \mathrm{Map}^{G/H}(X^H, Y^H).$$

The first equality follows because for any G-equivariant map $f : X^H \to Y$ and any $h \in H$ and $x \in X^H$ we have $h \cdot f(x) = f(h \cdot x) = f(x)$ since x is an H-fixed point, and hence f maps into the H-fixed points of Y. The second equality follows because the data of a G-equivariant map between the H-fixed points of spaces is exactly the data of a corresponding G/H-equivariant map.

As we alluded to above, we build the inclusion $X^H \cup X^{\not\le H} \to X$ inducing the map $\mathrm{Map}^G(X, Y) \to \mathrm{Map}^G(X^H \cup X^{\not\le H}, Y)$ by induction on the cells which much be successively attached starting with $X^H \cup X^{\not\le H}$. We proceed by induction; in the base case the identity on $X_0 = X^H \cup X^{\not\le H}$ is a weak equivalence and fibration, so there is nothing to show. Then given a cell attaching map $f : G/K_j \times S^{k_j} \to X_j$, the space obtained upon attaching along f is the pushout (i.e. span colimit)

$$\begin{array}{ccc} G/K_j \times S^{k_j} & \hookrightarrow & G/K_j \times D^{k_j+1} \\ \downarrow f & & \vdots \\ X_j & \cdots\cdots\to & X_{j+1} \end{array},$$

and so we can form a commuting diagram (noting that $\mathrm{Map}^G(-,Y)$ sends colimits to limits by Proposition 2.4)

$$\begin{array}{ccc}
\mathrm{Map}^G(S^{k_j}, Y^{K_j}) & & \mathrm{Map}^G(D^{k_j+1}, Y^{K_j}) \\
\downarrow \wr & & \downarrow \wr \\
\mathrm{Map}^G(G/K_j \times S^{k_j}, Y) & \leftarrow\cdots\cdots & \mathrm{Map}^G(G/K_j \times D^{k_j+1}, Y) \\
\vdots\,\mathrm{Map}^G(f,Y) & & \uparrow \\
\mathrm{Map}^G(X_j, Y) & \longleftarrow & \mathrm{Map}^G(X_{j+1}, Y)
\end{array}$$

using the isomorphisms of Lemma 4.6, with the dotted morphisms now part of a pullback (a limit of the lower cospan). But now the space Y^{K_j} is contractible by the hypothesis that Y is concentrated at $[H]$, so the top row of the square is a map between contractible objects.

Since the top map is therefore a weak equivalence, it now follows formally (model-categorically) by Lemma 2.3 that the bottom map is a weak equivalence as well. We can then form the finite composite $\mathrm{Map}^G(X,Y) \to \mathrm{Map}^G(X^H \cup X^{\not\leq H}, Y)$ of the individual attaching maps $\mathrm{Map}^G(X_{j+1},Y) \to \mathrm{Map}^G(X_j,Y)$, and conclude it is therefore in particular a fibration and weak equivalence.

We treat the restriction $\mathrm{Map}^G(X^H \cup X^{\not\leq H}, Y) \to \mathrm{Map}^G(X^H, Y)$ in a similar manner; being built from a point by attaching cells $G/K \times S^k$ with $K \not\leq H$, the subcomplex $X^{\not\leq H}$ and the intersection $X^H \cap X^{\not\leq H}$ are both weakly equivalent to a contractible space by precisely the same argument. In addition, under $\mathrm{Map}^G(-,Y)$ the pushout square

$$\begin{array}{ccc}
X^H \cap X^{\not\leq H} & \longrightarrow & X^H \\
\downarrow & & \downarrow \\
X^{\not\leq H} & \longrightarrow & X^H \cup X^{\not\leq H}
\end{array}$$

becomes the pullback

$$\begin{array}{ccc}
\mathrm{Map}^G(X^H \cap X^{\not\leq H}, Y) & \longleftarrow & \mathrm{Map}^G(X^H, Y) \\
\uparrow & & \uparrow \\
\mathrm{Map}^G(X^{\not\leq H}, Y) & \longleftarrow & \mathrm{Map}^G(X^H \cup X^{\not\leq H}, Y)
\end{array} .$$

We have just seen that the two spaces on the left are weakly equivalent to contractible spaces, so the map connecting them is a weak equivalence, and thus again by Lemma 2.3 we conclude that the map on the right (which we wanted to analyse) is in particular a weak equivalence in addition to being a fibration. Being the composite of maps simultaneously fibrations and weak equivalences ρ is therefore itself a fibration and weak equivalence, and this completes the proof. □

The following lemma is a completely trivial corollary of ordinary induction which is possible in the case of finite groups. Nonetheless, the idea is made very useful by Proposition 4.9 which follows it.

Lemma 4.8 (tom Dieck induction)**.** *Let P be a predicate on elements of $\mathscr{F}_G$. If for all $T, S \in \mathscr{F}_G$ with $T \subset S$ whenever $P(T)$ holds then $P(S)$ holds, then $P(\mathscr{C}_G)$ holds.*

Its formal triviality notwithstanding, this second notion of induction in equivariant stable homotopy theory (as opposed to induction in the cells of a G-CW-complex) was of great pedagogical importance when first introduced by tom Dieck, and for example was a key step in the resolution of the Kervaire invariant 1 problem [1]. Before we state and prove the proposition which makes this lemma interesting, we just recall the definition of a $\mathbf{Z}$-graded (co)homology theory on G-spaces[11]; this is the data of a family of homotopy-invariant functors $(h_n : G\mathsf{Top}_* \to \mathsf{Ab})_{n\in\mathbf{Z}}$ which send wedge sums to direct sums and possess the standard long exact sequence associated to every cofiber sequence.

Proposition 4.9 (Reduction to concentrated case)**.** *Let $h_\bullet$ and $h'_\bullet$ be $\mathbf{Z}$-graded (co)homology theories, and let $f : h_\bullet \to h'_\bullet$ be a natural transformation of them. Then f is an isomorphism if and only if for all $[H] \in \mathscr{C}_G$ and every X concentrated at $[H]$ the component f_X is an isomorphism.*

[11] Some standard properties are summarised in [12].

Proof. We only handle the case of homology theories, since the cohomological version is completely analogous; in any case the forward direction is immediate. For the other implication, fix any G-space X and define a predicate P on $\mathscr{F}_G$ by

$$P(S) = \text{the map } f_{ES_+\wedge X} : h_\bullet(ES_+ \wedge X) \to h'_\bullet(ES_+ \wedge X) \text{ is an isomorphism.}$$

When $S = \{\}$ then ES_+ is just the adjoined point which is concentrated (for example) at $e \le G$. Hence $P(\{\})$ holds. Otherwise, suppose that $P(T)$ holds for all $T \subset S$ for some $S \neq \emptyset$, and fix some T such that $S = T \cup \{[H]\}$. By the universal property of classifying spaces, there is a map $\iota : ET \to ES$ unique up to G-equivariant homotopy, and ι naturally fits into a cofiber sequence

$$ET_+ \to ES_+ \to C_\iota.$$

For any $K \le G$ with $[K] \notin T$ we have $(ET_+)^K = (ES_+)^K = *$, so necessarily $C_\iota^K \simeq *$ as well. When instead $[K] \in T$ and $[K] \neq [H]$ both ET and ES are contractible, so again C_ι is too. Smashing the sequence with X we obtain that

$$ET_+ \wedge X \to ES_+ \wedge X \to C_\iota \wedge X,$$

so again $C_\iota \wedge X$ is concentrated at $[H]$.

Hence by assumption the map $f_{C_\iota \wedge X} : h_\bullet(C_\iota \wedge X) \to h'_\bullet(C_\iota \wedge X)$ is an isomorphism. The long exact sequence of each homology theory for this cofiber sequence gives a diagram

$$\begin{array}{ccccccccccc} \cdots \to & h_{n+1}(C_\iota \wedge X) & \to & h_n(ET_+ \wedge X) & \to & h_n(ES_+ \wedge X) & \to & h_n(C_\iota \wedge X) & \to & h_{n-1}(ET_+ \wedge X) & \to \cdots \\ & \downarrow & & \downarrow & & \vdots & & \downarrow & & \downarrow & \\ \cdots \to & h'_{n+1}(C_\iota \wedge X) & \to & h'_n(ET_+ \wedge X) & \to & h'_n(ES_+ \wedge X) & \to & h'_n(C_\iota \wedge X) & \to & h'_{n-1}(ET_+ \wedge X) & \to \cdots \end{array}$$

We have just seen that the first and fourth maps are isomorphisms. Moreover, the second and fifth are isomorphisms by hypothesis (we have that $P(T)$ holds), so by the five lemma we conclude that $f_{ES_+\wedge X} : h_n(ES_+ \wedge X) \to h'_n(ES_+ \wedge X)$ is an isomorphism as well (all of the squares commute by naturality of f).

Appealing to Lemma 4.8 we conclude that the component

$$f_{E\mathscr{C}_{G+}\wedge X} : h_n(E\mathscr{C}_{G+} \wedge X) \to h'_n(E\mathscr{C}_{G+} \wedge X)$$

is an isomorphism. But $(E\mathscr{C}_G)^K$ is contractible for all $K \le G$, and hence there is a G-equivariant homotopy equivalence $E\mathscr{C}_{G+} \wedge X \to X$. Therefore again by naturality of f there is a commuting square

$$\begin{array}{ccc} h_\bullet(E\mathscr{C}_{G+} \wedge X) & \to & h_\bullet(X) \\ \downarrow & & \vdots \\ h'_\bullet(E\mathscr{C}_{G+} \wedge X) & \to & h'_\bullet(X) \end{array}$$

with three of the maps isomorphisms, so the (dotted) fourth is as well. Hence every component of f is an isomorphism. □

Proof of tom Dieck splitting (Theorem 4.4). The isomorphism of tom Dieck splitting is built by directly assembling individual maps from each summand (which themselves are certainly not isomorphisms in general). Namely, for each $[H] \in \mathscr{C}_G$ we need a map of homotopy groups

$$\pi_\bullet^{WH}(\Sigma^\infty(EWH_+ \wedge X^H)) \to \pi_\bullet^G(\Sigma^\infty X).$$

To do this, we can leverage the transfer map $T_{\mathrm{N}H}^G$ obtained in Proposition 3.5 via the Wirthmüller isomorphism, which itself is a morphism

$$\pi_\bullet^{\mathrm{N}H}(\Sigma^\infty(EWH_+ \wedge X)) \xrightarrow{T_{\mathrm{N}H}^G} \pi_\bullet^G(G \wedge_{\mathrm{N}H} \Sigma^\infty(EWH_+ \wedge X)).$$

In order to use this map to connect the homotopy groups in question, observe that there is a natural way to way to obtain a map from $\pi_\bullet^{WH}(\Sigma^\infty(EWH_+ \wedge X^H))$ to $\pi_\bullet^{\mathrm{N}H}(\Sigma^\infty(EWH_+ \wedge X))$; namely

$$\pi_\bullet^{WH}(\Sigma^\infty(EWH_+ \wedge X^H)) \xrightarrow{\pi_{\mathrm{N}H}} \pi_\bullet^{\mathrm{N}H}(\Sigma^\infty(EWH_+ \wedge X^H)) \xrightarrow{(X^H \hookrightarrow X)_*} \pi_\bullet^{\mathrm{N}H}(\Sigma^\infty(EWH_+ \wedge X)).$$

Here the first map is just the natural morphism $\pi_\bullet^{WH} \to \pi_\bullet^{NH}$ (i.e. induced since WH is the quotient of NH by H), and the second is induced by the inclusion ι_H of X^H into X.

Similarly, in order to connect the transfer map T^G_{NH} to $\pi^G_\bullet(\Sigma^\infty X)$ we use the natural map induced by $EWH \to *$, which just obliterates the EWH part, to obtain a map

$$\Sigma^\infty(EWH_+ \wedge X) \to \Sigma^\infty(X)$$

of NH-spectra. The adjunction of Proposition 2.5 then yields a map

$$G \wedge_{NH} \Sigma^\infty(EWH_+ \wedge X) \to \Sigma^\infty(X)$$

of G-spectra, which induces the desired

$$\xi_H : \pi^G_\bullet(G \wedge_{NH} \Sigma^\infty(EWH_+ \wedge X)) \to \pi^G_\bullet(\Sigma^\infty(X)).$$

Each of these composites assemble together to yield a map

$$D^G_X = \bigoplus_{[H]\in\mathscr{C}_G} \xi_H \circ T^G_{NH} \circ \iota_{H*} \circ \pi_{NH} : \bigoplus_{[H]\in\mathscr{C}_G} \pi^{WH}_\bullet(\Sigma^\infty(EWH_+ \wedge X^H)) \to \pi^G_\bullet(\Sigma^\infty X),$$

and showing that the result is always an isomorphism is intimidating. The key point is that by allowing X to vary in the construction above, the domain and codomain of D^G_X are both **Z**-graded homology theories, and hence Proposition 4.9 allows us to reduce to the case when X^H is contractible for all H not in a distinguished class $[K] \in \mathscr{C}_G$.

In this situation, all but one of the suspension spectra being summed over become contractible, so it will be sufficient to show that

$$\xi_K \circ T^G_{NK} \circ \iota_{K*} \circ \pi_{NK}$$

is an isomorphism. To see this, let $H \le G$ with $H \notin [K]$ and let V be any WH-representation. Then for any $H' \le$ WH we have that $EWH^{H'}$ is empty if H' is not the trivial group, and in any case X^H is contractible just because $H \notin [K]$ by definition. Hence regardless of whether $H' = e$ or not the H'-fixed points of the space $S^V \wedge EWH_+ \wedge X^H$ are contractible. Therefore the V-piece of $\Sigma^\infty(EWH_+ \wedge X^H)$ is weakly equivalent to a contractible space, and thus by the definition of the homotopy groups $\pi_\bullet(\Sigma^\infty(EWH_+ \wedge X^H))$ this ensures that each vanishes.

Since T^G_{NK} is already an isomorphism, this means that we just need to establish that ξ_K and the composite $\iota_{K*} \circ \pi_{NK}$ are both isomorphisms as well (individually ι_{K*} and π_{NK} are not isomorphisms in general). Thus, it remains to establish the following pair of lemmas.

Lemma 4.10. *When X is concentrated at $[H]$, the map*

$$\xi_H : \pi^G_\bullet(G \wedge_{NH} \Sigma^\infty(EWH_+ \wedge X)) \to \pi^G_\bullet(\Sigma^\infty(X))$$

is an isomorphism.

Proof. Let $K \le G$ be arbitrary, and let V be a G-representation. The map on the V-piece of the orthogonal spectra in question is a map

$$G \wedge_{NH} (S^V \wedge EWH_+ \wedge X) \to S^V \wedge X,$$

and thus it is sufficient to show that for each V and $K \le G$ the induced map of K-fixed points is a weak equivalence. We start by rewriting $G \wedge_{NH} (S^V \wedge EWH_+ \wedge X)$ as $S^V \wedge (G \wedge_{NH} EWH)_+ \wedge X$ using the fact that S^V and X are both G-spaces. Now taking the K-fixed points we can distribute over the wedge sum of ordinary G-spaces and instead interpret the V-piece of ξ_H as a map (certainly the operations of taking K-fixed points and adjoining a basepoint via "+" commute)

$$\xi^K_{H,V} : (S^V)^K \wedge ((G \wedge_{NH} EWH)^K)_+ \wedge X^K \to (S^V)^K \wedge X^K.$$

Note also that since ξ_H is the adjoint of the map induced by sending EWH to a point, the map $\xi^K_{H,V}$ is just the projection which forgets the second wedge summand.

First consider the case when $K \in [H]$. Then since in $(G \wedge_{NH} EWH)_+$ the wedge is being taken over NH, we have $(G \wedge_{NH} EWH)^K \cong (EWH)^K$. But now $(EWH)^K$ is contractible by the definition of the universal space EWH since $K \in [H]$, so $\xi^K_{H,V}$ is the desired weak equivalence. On the other hand X is concentrated at $[H]$ by hypothesis, so whenever $K \notin [H]$ we have that X^K is contractible and again $\xi^K_{H,V}$ is automatically a weak equivalence. This exhausts all possible cases, so we conclude that ξ_H is a weak equivalence, as desired. □

Lemma 4.11. *When X is concentrated at $[H]$, the composite*

$$\iota_{H*}\circ\pi_{\mathrm{N}H}:\pi_\bullet^{\mathrm{W}H}(\Sigma^\infty(E\mathrm{W}H_+\wedge X^H))\to\pi_\bullet^{\mathrm{N}H}(\Sigma^\infty(E\mathrm{W}H_+\wedge X))$$

is an isomorphism.

Proof. It is sufficient to give a map ϕ which induces a one-sided and bijective inverse ϕ_* of the map of homotopy groups $\psi_*=\iota_{H*}\circ\pi_{\mathrm{N}H}$. By definition, for $n\geq 0$ we have (the argument for the $n<0$ case is essentially identical)

$$\begin{aligned}\pi_n^{\mathrm{N}H}(\Sigma^\infty(E\mathrm{W}H_+\wedge X))&=\mathrm{colim}_V\,\pi_n^{\mathrm{N}H}\Omega^V(S^V\wedge E\mathrm{W}H_+\wedge X)\\&=\mathrm{colim}_V[S^V\wedge S^n,S^V\wedge E\mathrm{W}H_+\wedge X]^{\mathrm{N}H},\end{aligned}$$

so by taking $G=\mathrm{N}H$ in Proposition 4.7 (note that in the proposition H need not be normal in all of G, but is necessarily normal in $\mathrm{N}H$, and observe that $S^V\wedge S^n$ is a G-CW-complex with finitely many cells since V is finite dimensional) we obtain for each V a bijection

$$[S^V\wedge S^n,S^V\wedge E\mathrm{W}H_+\wedge X]^{\mathrm{N}H}\to[S^V\wedge S^n,(S^V)^H\wedge(E\mathrm{W}H)_+^H\wedge X^H]^{\mathrm{N}H/H}.$$

Now, by definition $\mathrm{N}H/H=\mathrm{W}H$, and as we saw in the previous part H leaves $E\mathrm{W}H$ invariant, so we really have a bijection

$$[S^V\wedge S^n,S^V\wedge E\mathrm{W}H_+\wedge X]^{\mathrm{N}H}\to[(S^V)^H\wedge S^n,(S^V)^H\wedge E\mathrm{W}H_+\wedge X^H]^{\mathrm{W}H}.$$

By cofinality of the sequence of regular representations of Lemma 2.1 it suffices to assume $V=kR=\bigoplus_{i=1}^k R$ for R the regular representation of $\mathrm{N}H$. Since $(S^V)^H\cong S^{V^H}$ and $(kR)^H\cong kR'$ for R' the regular representation of $\mathrm{W}H$ (this latter isomorphism obviously does not hold in general), we obtain a bijection

$$[S^{kR}\wedge S^n,S^{kR}\wedge E\mathrm{W}H_+\wedge X]^{\mathrm{N}H}\to[S^{kR'}\wedge S^n,S^{kR'}\wedge E\mathrm{W}H_+\wedge X^H]^{\mathrm{W}H}.$$

These maps hence assemble in the colimit defining the stable homotopy groups to induce an isomorphism

$$\phi_*:\pi_\bullet^{\mathrm{N}H}(\Sigma^\infty(E\mathrm{W}H_+\wedge X))\to\pi_\bullet^{\mathrm{W}H}(\Sigma^\infty(E\mathrm{W}H_+\wedge X^H)).$$

Since it is clear that the map ϕ_* is a post-inverse of ψ_*—in particular elements of the image of ψ_* already have H acting by the identity—this completes the proof. □

With this lemma established, the proof of tom Dieck splitting is complete. □

5 Generalisations and applications

Both the tom Dieck splitting theorem and the Wirthmüller isomorphism are fundamental structural results regarding the category of G-spectra and its homotopy category. All of our results generalise (naturally, though occasionally with some modification) to the case of G a compact Lie group (in this case [11] is a good reference). In a different vein, Theorem 4.1 of [3] develops a generalised tom Dieck splitting for arbitrary homotopy functors which commute with fixed points. On the other hand [10] gives a treatment of the general category-theoretic framework into which the Wirthmüller and similar Grothendieck isomorphisms fit (in particular, into Grothendieck's yoga of six functors).

We conclude with a very concrete application of Theorem 4.4; setting $X=\mathbb{S}$ the sphere spectrum, tom Dieck splitting gives a formula

$$\pi_0^G(\mathbb{S})\cong\bigoplus_{[H]\in\mathscr{C}_G}\pi_0^{\mathrm{W}H}(\Sigma^\infty(E\mathrm{W}H_+))$$

for the zeroth equivariant stable homotopy group of the sphere spectrum. Hence $\pi_0^G(\mathbb{S})$ is the free abelian group on the conjugacy classes of subgroups of G. It is a theorem[12] of Segal [13] that this map can be upgraded into a ring isomorphism from the Burnside ring $A(G)$ to $\pi_0^G(\mathbb{S})$ which commutes with the transfer and restriction maps associated to group homomorphisms.

[12] A modern account of the proof can be found in [12].

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